COMPUTATION OF ELLIPSOIDAL GRAVITY FIELD HARMONICS FOR SMALL SOLAR SYSTEM BODIES

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Master of Science
(Electrical Engineering - Systems)
in The University of Michigan
2000

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Acknowledgments

To begin with I would like to express my gratitude to my advisor Professor Daniel Scheeres for offering me the chance to work with him on the NASA NEAR mission and introducing me to graduate research. I would also like to express my appreciation for his continual support, guidance and availability. I have been very pleased with this research work since it involved such a challenging mission as the NEAR mission, we achieved the goals that we had set and it made my stay at the University of Michigan such a wonderful and valuable experience.

I would like to thank Professor Pierre Kabamba and Professor Harris McClamroch for consenting to be on my thesis committee and expressing their interest in the subject. I would also like to thank Heidi Di Virgilio and my office mates, Alexander Roup, Harshad Sane, Daniel Scharf and Biju Thuruthimattam for their help in my thesis research and writing.

Contents

Introduction

Small solar system bodies like asteroids and comets have become the target of current and forthcoming space missions. The NEAR Shoemaker spacecraft has been in orbit around Asteroid 433 Eros since February 14th, 2000. The MUSES-C mission is a joint effort of Japan's Institute of Space and Astronautical Sciences (ISAS) and NASA to explore Asteroid 1998 SF36 and return samples from the asteroid surface to Earth. It is scheduled for launch in December 2002 and is expected to arrive on target in September 2005. Finally, the ESA Rosetta mission's main objective is to rendezvous with and make insitu measurements of comet 46 P/Wirtanen, in August 2011. These bodies are known to have low gravity and irregular shapes which can cause severe problems for predicting a spacecraft trajectory in their vicinity.

The classical approach for representing the gravitational field of an arbitrary body is by expanding it's gravitational potential into spherical harmonics. The density and shape of the body is modeled by coefficients known as spherical harmonic coefficients C_{lm} and S_{lm}. These coefficients can be determined with high accuracy by evaluating the perturbations induced by the body on the spacecraft trajectory. The advantage of this method is that it involves simple mathematics and converges to the correct gravity field outside a circumscribing sphere. In addition, finite truncation orders are often sufficient to match the "true potential" of the body with good accuracy. For Eros (of approximate size 33 km $\times$ 13 km $\times$ 13 km), a 16 degree and order expansion is used to model the gravitational potential at a radius of 35 km. One of the major drawback of the spherical harmonics expansion is that it can exhibit severe divergence inside the circumscribing sphere. Thus, spherical harmonics expansions of the potential can not be used to model the asteroids' gravity at close range.

Another approach consists of modeling the asteroid gravity field as a constant density polyhedron. Small details of the asteroid surface can be included by modeling these regions with high resolution. The main advantage of this method is that the polyhedral potential is valid and exact for any given shape and density up to the surface of the body. Errors are thus reduced entirely to the errors in the asteroid shape detemination and the level of discretization chosen for that shape. For Asteroid 433 Eros, a polyhedron model with 8200 plates was used for the purpose of close flybys of the surface. Despite the exactitude of the polyhedral potential, the constant density assumption can lead to erroneous gravity computations at close range. Improvements to match the "true gravity field" more closely can be made by simulating density variations within the asteroid.

Gravity field computations are practically computed by performing a transition from the spherical harmonics expansion model to the polyhedron model as the spacecraft moves closer to the surface. The limitation due to the constant density assumption to accurately compute the polyhedral potential at close range motivated our work. The objective of this thesis is to develop a method based on ellipsoidal harmonics that matches the "true gravity field" in the vicinity of the asteroid's surface. In the first Chapter, we first present the theory of ellipsoidal harmonics expansion, introducing the ellipsoidal coordinates system and the Lamé functions. This theory was first developed by Hobson [9], MacMillan [11] and Byerly [3]. In Chapter 2, the focus is given to a transformation from the spherical harmonics coefficients C_{lm} and S_{lm} to the ellipsoidal harmonics coefficients α_{np}. New results on the theory of ellipsoidal harmonics are presented. Chapter 3 adresses the practical use of the ellipsoidal harmonics describing numerical methods related to the mathematical concepts of Chapters 1 and 2. Numerous references are made to Garmier [8] and Ritter [14]. Finally, Chapter 4 is dedicated to some numerical computations of the ellipsoidal harmonics expansion of the potential intented to validate the new approach developed in Chapter 2.

Chapter 1

Theory of ellipsoidal harmonics expansion

1.1 Definition of the ellipsoidal coordinates

Define the fundamental ellipsoid as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \text{with} \quad a > b > c > 0, \tag{1.1}$$

and any confocal quadric as

$$\frac{x^2}{a^2+\theta^2} + \frac{y^2}{b^2+\theta^2} + \frac{z^2}{c^2+\theta^2} = 1. \tag{1.2}$$

Let us perform the following change of variables $\lambda^2 = a^2 + \theta^2$ and define

$$\begin{cases} h^2 & = a^2 - b^2 \\ k^2 & = a^2 - c^2. \end{cases} \tag{1.3}$$

Then (1.2) becomes

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2 - h^2} + \frac{z^2}{\lambda^2 - k^2} = 1. \tag{1.4}$$

For x, y and z fixed, the preceding equation has three roots ${\lambda_1}^2$, ${\lambda_2}^2$ and ${\lambda_3}^2$ such that:

$${\lambda_1}^2 > k^2 > {\lambda_2}^2 > h^2 > {\lambda_3}^2 > 0.$$

The parameters $(\lambda_1, \lambda_2, \lambda_3)$ are called the ellipsoidal coordinates. $\lambda_1 = \mathit{constant}$ is the equation of a triaxial ellipsoid of semi-axes $(\lambda_1, \sqrt{\lambda_1^2 - h^2}, \sqrt{\lambda_1^2 - k^2})$. For a given (h,k), the family of ellipsoids $\lambda_1 = constant$ is homofocal to the reference ellispoid (1.1). By analogy with spherical coordinates, λ_1 is called the "elliptic radius".

One can relate the ellipsoidal coordinates to the cartesian coordinates as follows:

$$x^2 = \frac{\lambda_1^2\lambda_2^2\lambda_3^2}{h^2k^2} \quad (1.5)$$

$$y^2 = \frac{(\lambda_1^2-h^2)(\lambda_2^2-h^2)(h^2-\lambda_3^2)}{h^2(k^2-h^2)} \quad (1.6)$$

$$z^2 = \frac{(\lambda_1^2-k^2)(k^2-\lambda_2^2)(k^2-\lambda_3^2)}{k^2(k^2-h^2)}. \quad (1.7)$$

There are eight points corresponding to the same $(\lambda_i^2)_{i=1,2,3}$. These points are at the intersection of the ellipsoid defined by λ_1^2, the hyperboloid of one sheet defined by λ_2^2 and the hyperboloid of two sheets defined by λ_3^2 (see Figures 1.1 and 1.2). In order that each point (x,y,z) may be expressed by a single set of values $(\lambda_1,\lambda_2,\lambda_3)$, we impose:

- λ_3 to be taken with the $\begin{cases} + \text{ sign for } x \geq 0 \\ - \text{ sign for } x < 0 \end{cases}$

- $\sqrt{h^2-\lambda_3^2}$ to be taken with the $\begin{cases} + \text{ sign for } y \geq 0 \\ - \text{ sign for } y < 0 \end{cases}$

- $\sqrt{k^2-\lambda_2^2}$ to be taken with the $\begin{cases} + \text{ sign for } z \geq 0 \\ - \text{ sign for } z < 0 \end{cases}$

- λ_1, $\sqrt{\lambda_1^2-h^2}$, $\sqrt{\lambda_1^2-k^2}$, λ_2, $\sqrt{\lambda_2^2-h^2}$, and $\sqrt{k^2-\lambda_3^2}$ to be taken always with the $+$ sign.

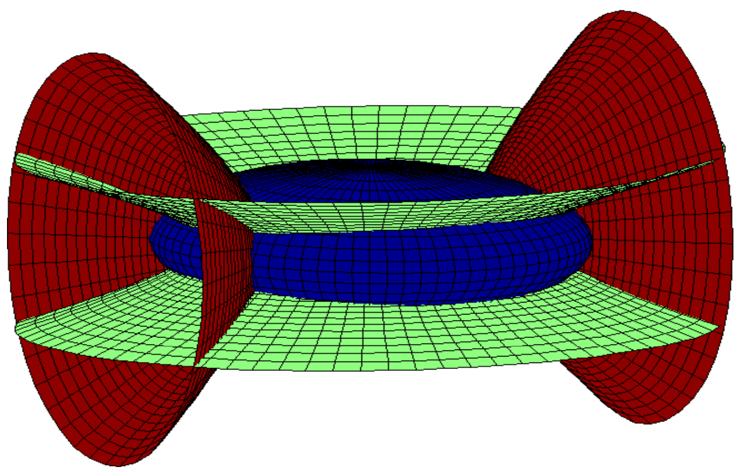


Figure 1.1: Intersection of the 3 surfaces defined by λ_1^2, λ_2^2 and λ_3^2

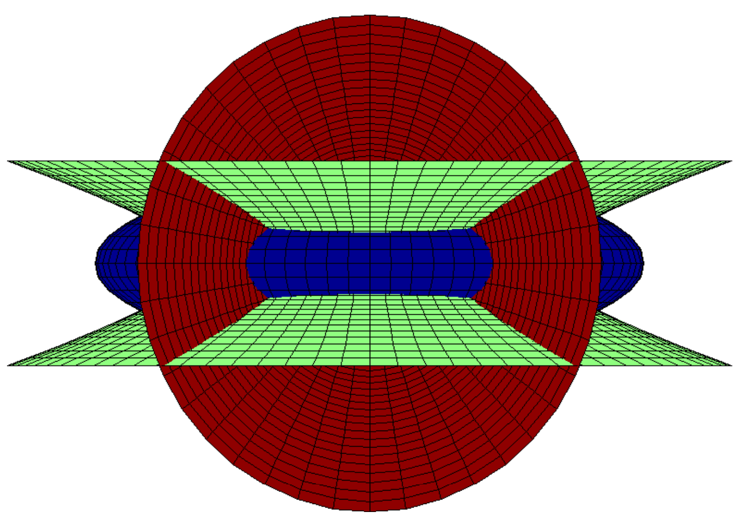


Figure 1.2: 4 of the 8 intersection points

1.2 Laplace's equation in terms of the ellipsoidal coordinates

In order to obtain solutions of Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{1.8}$$

appropriate for spaces with boundaries of various forms, it is convenient to transform the equation into a form in which the independent variables are the parameters of three orthogonal sets of surfaces. In the following, we will rewrite Laplace's equation (1.8) in terms of the ellipsoidal coordinates.

The length of the elementary line joining the two points (x, y, z) and $(x+dx, y+dy, z+dz)$ is given by

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2. \tag{1.9}$$

Differentiating (1.5), (1.6) and (1.7), $(ds)^2$ can be expressed in terms of $(\lambda_1, \lambda_2, \lambda_3)$. This yields

$$(ds)^2 = \frac{1}{H_1^2}(d\lambda_1)^2 + \frac{1}{H_2^2}(d\lambda_2)^2 + \frac{1}{H_3^2}(d\lambda_3)^2 \tag{1.10}$$

where

$$H_1^2 = \frac{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)}{(\lambda_1^2 - h^2)(\lambda_1^2 - k^2)} \tag{1.11}$$

$$H_2^2 = \frac{(\lambda_2^2 - \lambda_1^2)(\lambda_2^2 - \lambda_3^2)}{(\lambda_2^2 - h^2)(\lambda_2^2 - k^2)} \tag{1.12}$$

$$H_3^2 = \frac{(\lambda_3^2 - \lambda_2^2)(\lambda_3^2 - \lambda_1^2)}{(\lambda_3^2 - h^2)(\lambda_3^2 - k^2)}. \tag{1.13}$$

In the orthogonal ellipsoidal coordinates $(\lambda_1, \lambda_2, \lambda_3)$, Laplace's equation $\nabla^2 V = 0$ takes the form

$$\frac{\partial}{\partial\lambda_1}\left(\frac{H_1}{H_2H_3}\frac{\partial V}{\partial\lambda_1}\right) + \frac{\partial}{\partial\lambda_2}\left(\frac{H_2}{H_3H_1}\frac{\partial V}{\partial\lambda_2}\right) + \frac{\partial}{\partial\lambda_3}\left(\frac{H_3}{H_1H_2}\frac{\partial V}{\partial\lambda_3}\right) = 0. \tag{1.14}$$

Computing

$$\frac{H_1}{H_2H_3} = (\lambda_2^2 - \lambda_3^2)\sqrt{\frac{(\lambda_1^2 - h^2)(\lambda_1^2 - k^2)}{(\lambda_2^2 - h^2)(k^2 - \lambda_2^2)(h^2 - \lambda_3^2)(k^2\lambda_3^2)}} \tag{1.15}$$

$$\frac{H_2}{H_3H_1} = (\lambda_1^2 - \lambda_3^2)\sqrt{\frac{(\lambda_2^2 - h^2)(k^2 - \lambda_2^2)}{(\lambda_1^2 - h^2)(\lambda_1^2 - k^2)(h^2 - \lambda_3^2)(k^2 - \lambda_3^2)}} \tag{1.16}$$

$$\frac{H_3}{H_1H_2} = (\lambda_1^2-\lambda_2^2)\sqrt{\frac{(h^2-\lambda_3^2)(k^2-\lambda_3^2)}{(\lambda_1^2-h^2)(\lambda_1^2-k^2)(\lambda_2^2-h^2)(k^2-\lambda_2^2)}} \quad (1.17)$$

and replacing the above expressions into (1.14), we obtain after multiplying both sides by $\sqrt{(\lambda_1^2-h^2)(\lambda_1^2-k^2)(\lambda_2^2-h^2)(k^2-\lambda_2^2)(h^2-\lambda_3^2)(k^2-\lambda_3^2)}$:

$$\left.\begin{array}{l}(\lambda_2^2-\lambda_3^2)\sqrt{(\lambda_1^2-h^2)(\lambda_1^2-k^2)}\,\dfrac{\partial}{\partial\lambda_1}\left(\sqrt{(\lambda_1^2-h^2)(\lambda_1^2-k^2)}\,\dfrac{\partial V}{\partial\lambda_1}\right)\\ +(\lambda_1^2-\lambda_3^2)\sqrt{(\lambda_2^2-h^2)(k^2-\lambda_2^2)}\,\dfrac{\partial}{\partial\lambda_2}\left(\sqrt{(\lambda_2^2-h^2)(k^2-\lambda_2^2)}\,\dfrac{\partial V}{\partial\lambda_2}\right)\\ +(\lambda_1^2-\lambda_2^2)\sqrt{(h^2-\lambda_3^2)(k^2-\lambda_3^2)}\,\dfrac{\partial}{\partial\lambda_3}\left(\sqrt{(h^2-\lambda_3^2)(k^2-\lambda_3^2)}\,\dfrac{\partial V}{\partial\lambda_3}\right)\end{array}\right\}=0. \quad (1.18)$$

We are looking for solutions of the form

$$V(\lambda_1,\lambda_2,\lambda_3) = R(\lambda_1)M(\lambda_2)N(\lambda_3). \quad (1.19)$$

Such a V is called a normal solution. Laplace's equation (1.18) can then be rewritten as

$$(\lambda_2^2-\lambda_3^2)\,\phi_1(\lambda_1) + (\lambda_3^2-\lambda_1^2)\,\phi_2(\lambda_2) + (\lambda_1^2-\lambda_2^2)\,\phi_3(\lambda_3) = 0 \quad (1.20)$$

where

$$\phi_1(\lambda_1) = \frac{\sqrt{(\lambda_1^2-h^2)(\lambda_1^2-k^2)}}{R(\lambda_1)}\,\frac{d}{d\lambda_1}\left(\sqrt{(\lambda_1^2-h^2)(\lambda_1^2-k^2)}\,\frac{dR}{d\lambda_1}\right) \quad (1.21)$$

$$\phi_2(\lambda_2) = -\frac{\sqrt{(\lambda_2^2-h^2)(k^2-\lambda_2^2)}}{M(\lambda_2)}\,\frac{d}{d\lambda_2}\left(\sqrt{(\lambda_2^2-h^2)(k^2-\lambda_2^2)}\,\frac{dM}{d\lambda_2}\right) \quad (1.22)$$

$$\phi_3(\lambda_3) = \frac{\sqrt{(h^2-\lambda_3^2)(k^2-\lambda_3^2)}}{N(\lambda_3)}\,\frac{d}{d\lambda_3}\left(\sqrt{(h^2-\lambda_3^2)(k^2-\lambda_3^2)}\,\frac{dN}{d\lambda_3}\right). \quad (1.23)$$

Equation (1.20) holds for any $(\lambda_1,\lambda_2,\lambda_3)$. Taking $\lambda_2=\lambda_3$ first yields $\phi_2(\lambda_2) = \phi_3(\lambda_2)$, then $\phi_1(\lambda_1)=\phi_2(\lambda_1)=\phi_3(\lambda_1)=\phi(\lambda_1)$.
To determine the function ϕ, take $\lambda_1=0$ and rewrite (1.20) as

$$\frac{\phi(\lambda_2)-\phi(0)}{\lambda_2^2} = \frac{\phi(\lambda_3)-\phi(0)}{\lambda_3^2}. \quad (1.24)$$

Since (1.24) holds for any λ_2 and λ_3, it results that both sides of (1.24) have to be constant. The function ϕ is then of the form

$$\phi(\lambda_i) = H\lambda_i^2 - K \quad (1.25)$$

where H and K are constants.

Replacing ϕ into (1.21) yields Lamé's equation

$$\sqrt{(\lambda_1^2-h^2)(\lambda_1^2-k^2)}\,\frac{d}{d\lambda_1}\left(\sqrt{(\lambda_1^2-h^2)(\lambda_1^2-k^2)}\,\frac{dR}{d\lambda_1}\right) = (H\lambda_1^2-K)R(\lambda_1), \quad (1.26)$$

that can be rewritten as

$$(\lambda_1^2-h^2)(\lambda_1^2-k^2)\frac{d^2R}{d\lambda_1^2} + \lambda_1(2\lambda_1^2-h^2-k^2)\frac{dR}{d\lambda_1} + (K-H\lambda_1^2)R = 0. \quad (1.27)$$

M and N satisfy exactly the same type of equation as R replacing λ_1 respectively by λ_2 and λ_3:

$$(\lambda_2^2-h^2)(\lambda_2^2-k^2)\frac{d^2M}{d\lambda_2^2} + \lambda_2(2\lambda_2^2-h^2-k^2)\frac{dM}{d\lambda_2} + (K-H\lambda_2^2)M = 0 \quad (1.28)$$

$$(\lambda_3^2-h^2)(\lambda_3^2-k^2)\frac{d^2N}{d\lambda_3^2} + \lambda_3(2\lambda_3^2-h^2-k^2)\frac{dN}{d\lambda_3} + (K-H\lambda_3^2)N = 0. \quad (1.29)$$

Note that H and K remain the same as for R.

1.3 Lamé's functions of the first kind

Lamé's equation has the general form:

$$(\lambda_i^2-h^2)(\lambda_i^2-k^2)\frac{d^2E(\lambda_i)}{d\lambda_i^2} + \lambda_i(2\lambda_i^2-h^2-k^2)\frac{dE(\lambda_i)}{d\lambda_i} + (K-H\lambda_i^2)E(\lambda_i) = 0. \quad (1.30)$$

H and K are called the separation parameters and can be chosen so that the solutions of (1.30) are of four different types:

$$K_n^p(\lambda_i) = a_{0p}\lambda_i^n + a_{1p}\lambda_i^{n-2} + \ldots + \begin{cases} a_{\sigma p} & \text{for } n \text{ even} \\ a_{\sigma p}\lambda_i & \text{for } n \text{ odd} \end{cases} \quad (1.31)$$

$$L_n^p(\lambda_i) = \sqrt{|\lambda_i^2-h^2|} \times \left[b_{0p}\lambda_i^{n-1} + b_{1p}\lambda_i^{n-3} + \ldots + \begin{cases} b_{n-\sigma-1,p}\lambda_i & \text{for } n \text{ even} \\ b_{n-\sigma-1,p} & \text{for } n \text{ odd} \end{cases} \right] \quad (1.32)$$

$$M_n^p(\lambda_i) = \sqrt{|\lambda_i^2-k^2|} \times \left[c_{0p}\lambda_i^{n-1} + c_{1p}\lambda_i^{n-3} + \ldots + \begin{cases} c_{n-\sigma-1,p}\lambda_i & \text{for } n \text{ even} \\ c_{n-\sigma-1,p} & \text{for } n \text{ odd} \end{cases} \right] \quad (1.33)$$

$$N_n^p(\lambda_i) = \sqrt{|(\lambda_i^2-h^2)(\lambda_i^2-k^2)|} \times \left[d_{0p}\lambda_i^{n-2} + d_{1p}\lambda_i^{n-4} + \ldots + \begin{cases} d_{\sigma-1,p} & \text{for } n \text{ even} \\ d_{\sigma-1,p}\lambda_i & \text{for } n \text{ odd} \end{cases} \right] \quad (1.34)$$

with

$$\sigma = \begin{cases} \frac{1}{2}\,n & \text{for } n \text{ even,} \\ \frac{1}{2}\,(n-1) & \text{for } n \text{ odd.} \end{cases} \tag{1.35}$$

The functions above are called Lamé's functions of the first kind, of degree n and order p, and are denoted E_n^p. For a given n, the E_n^p are shared as follows:

- $(\sigma+1)$ are of type K, $p = 1, \ldots, (\sigma+1)$
- $(n-\sigma)$ are of type L, $p = (\sigma+2), \ldots, (n+1)$
- $(n-\sigma)$ of type M, $p = (n+2), \ldots, (2n-\sigma+1)$
- σ of type N, $p = (2n-\sigma+2), \ldots, (2n+1)$.

In total, there are $(2n+1)$ Lamé's functions of the first kind.

The Lamé product $E_n^p(\lambda_1)E_n^p(\lambda_2)E_n^p(\lambda_3)$ is a normal solution (e.g. (1.19)) to Laplace's equation (1.18) and is continuous within any ellipsoid $\lambda_1 = \lambda_1^{ref}$. It is then possible to define the potential for the interior space of the ellipsoid $\lambda_1 = \lambda_1^{ref}$ as:

$$V(\lambda_1, \lambda_2, \lambda_3) = \mu \sum_{n=0}^{\infty} \sum_{p=1}^{2n+1} \alpha_{np} \frac{E_n^p(\lambda_1)}{E_n^p(\lambda_1^{ref})} E_n^p(\lambda_2) E_n^p(\lambda_3), \quad \lambda_1 \leq \lambda_1^{ref}, \tag{1.36}$$

where the α_{np} are constants. The purpose of the division by $E_n^p(\lambda_1^{ref})$ will be explained in section 1.4.

1.4 Lamé's functions of the second kind

The Lamé product $E_n^p(\lambda_1)E_n^p(\lambda_2)E_n^p(\lambda_3)$ is however unbounded for $\lambda_1 \geq \lambda_1^{ref}$ because $E_n^p(\lambda_1) \to \infty$ when $\lambda_1 \to \infty$. The Lamé product $E_n^p(\lambda_1)E_n^p(\lambda_2)E_n^p(\lambda_3)$ is thus not acceptable to describe the potential for the exterior space of the ellipsoid $\lambda_1 = \lambda_1^{ref}$. A feasible normal solution to Laplace's equation should vanish when $\lambda_1 \to \infty$.

Let $F_n^p(\lambda_1)$ be a solution to Lamé's equation (1.26) that vanishes when $\lambda_1 \to \infty$ so that the normal solution $V = F_n^p(\lambda_1)E_n^p(\lambda_2)E_n^p(\lambda_3)$ is acceptable for the potential. Following Hobson's development (see [9]), $E_n^p(\lambda_1)$ and $F_n^p(\lambda_1)$ both satisfy equation (1.26). Thus

$$\frac{1}{E_n^p(\lambda_1)}\frac{d}{d\lambda_1}\left(\sqrt{(\lambda_1^2-h^2)(\lambda_1^2-k^2)}\,\frac{dE_n^p(\lambda_1)}{d\lambda_1}\right) = \frac{1}{F_n^p(\lambda_1)}\frac{d}{d\lambda_1}\left(\sqrt{(\lambda_1^2-h^2)(\lambda_1^2-k^2)}\,\frac{dF_n^p(\lambda_1)}{d\lambda_1}\right), \tag{1.37}$$

which is equivalent to

$$\frac{d}{d\lambda_1}\left[\sqrt{(\lambda_1^2-h^2)(\lambda_1^2-k^2)}\left(F_n^p(\lambda_1)\frac{dE_n^p(\lambda_1)}{d\lambda_1} - E_n^p(\lambda_1)\frac{dF_n^p(\lambda_1)}{d\lambda_1}\right)\right] = 0. \tag{1.38}$$

Thus

$$F_n^p(\lambda_1)\frac{dE_n^p(\lambda_1)}{d\lambda_1} - E_n^p(\lambda_1)\frac{dF_n^p(\lambda_1)}{d\lambda_1} = \frac{C}{\sqrt{(\lambda_1^2-h^2)(\lambda_1^2-k^2)}} \tag{1.39}$$

where C is a constant. Integrating between λ_1 and ∞ and choosing F_n^p such that $F_n^p(\infty) = 0$ yields:

$$\frac{F_n^p(\lambda_1)}{E_n^p(\lambda_1)} = C\int_{\lambda_1}^{\infty}\frac{du}{(E_n^p(\lambda_1))^2\sqrt{(u^2-k^2)(u^2-h^2)}}. \tag{1.40}$$

Hobson chooses $C = (2n+1)$ so that

$$\frac{F_n^p(\lambda_1)}{E_n^p(\lambda_1)} = \frac{1}{\lambda_1^{2n+1}} \quad \text{when } \lambda_1 \text{ is very large.} \tag{1.41}$$

With this choice of C, we finally get

$$F_n^p(\lambda_1) = (2n+1)\,E_n^p(\lambda_1)\int_{\lambda_1}^{\infty}\frac{du}{(E_n^p(u))^2\sqrt{(u^2-k^2)(u^2-h^2)}} \tag{1.42}$$

which is the required Lamé's function of the second kind of degree n and order p.

The Lamé product $F_n^p(\lambda_1)E_n^p(\lambda_2)E_n^p(\lambda_3)$ is a normal solution to Laplace's equation (1.18). In addition, it is continuous for $\lambda_1 \geq \lambda_1^{ref}$ and it vanishes when $\lambda_1 \to \infty$. It is then possible to define the potential outside the ellipsoid $\lambda_1 = \lambda_1^{ref}$ as:

$$V(\lambda_1,\lambda_2,\lambda_3) = \mu\sum_{n=0}^{\infty}\sum_{p=1}^{2n+1}\alpha_{np}\frac{F_n^p(\lambda_1)}{F_n^p(\lambda_1^{ref})}E_n^p(\lambda_2)E_n^p(\lambda_3), \quad \lambda_1 \geq \lambda_1^{ref}. \tag{1.43}$$

On the ellipsoid $\lambda_1 = \lambda_1^{ref}$, (1.36) and (1.43) have identical expressions so that the overall potential ($\lambda_1 \leq \lambda_1^{ref}$ and $\lambda_1 \geq \lambda_1^{ref}$) is continuous.

1.5 Normalized surface ellipsoidal harmonics

1.5.1 Orthogonalization property

Theorem 1 (Green) *Let S be any closed surface, and U_1 and U_2 any two harmonic functions. Then by Green's theorem,*

$$\int_S \left(U_2 \frac{\partial U_1}{\partial n} - U_1 \frac{\partial U_2}{\partial n} \right) dS = 0$$

where $\frac{\partial}{\partial n}$ is the normal derivative.

Let us consider the ellipsoid E_{λ_1}. The normal elementary displacement to the surface of the ellipsoid is

$$dn = \frac{1}{H_1} d\lambda_1. \tag{1.44}$$

Applying Green's theorem to the two harmonic functions

$$\begin{aligned} U_1 &= F_n^p(\lambda_1) E_n^p(\lambda_2) E_n^p(\lambda_3) & (1.45) \\ U_2 &= F_{n'}^{p'}(\lambda_1) E_{n'}^{p'}(\lambda_2) E_{n'}^{p'}(\lambda_3) & (1.46) \end{aligned}$$

on the surface of the ellipsoid yields:

$$\sqrt{(\lambda_1^2 - h^2)(\lambda_1^2 - k^2)} \left(F_{n'}^{p'}(\lambda_1) \frac{dF_n^p(\lambda_1)}{d\lambda_1} - F_n^p(\lambda_1) \frac{dF_{n'}^{p'}(\lambda_1)}{d\lambda_1} \right) \times \int_{E_{\lambda_1}} \frac{E_n^p(\lambda_2) E_n^p(\lambda_3) E_{n'}^{p'}(\lambda_2) E_{n'}^{p'}(\lambda_3)}{\sqrt{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)}} dS = 0. \tag{1.47}$$

Then, whether

$$F_{n'}^{p'}(\lambda_1) \frac{dF_n^p(\lambda_1)}{d\lambda_1} - F_n^p(\lambda_1) \frac{dF_{n'}^{p'}(\lambda_1)}{d\lambda_1} = 0, \tag{1.48}$$

or

$$\int_{E_{\lambda_1}} \frac{E_n^p(\lambda_2) E_n^p(\lambda_3) E_{n'}^{p'}(\lambda_2) E_{n'}^{p'}(\lambda_3)}{\sqrt{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)}} dS = 0. \tag{1.49}$$

Equation (1.48) is equivalent to

$$\frac{F_n^p(\lambda_1)}{F_{n'}^{p'}(\lambda_1)} = C, \tag{1.50}$$

where C is a constant. It arises if and only if $n = n'$ and $p = p'$.

Hence,

$$n \neq n' \text{ or } p \neq p' \quad \Rightarrow \quad \int_{E_{\lambda_1}} \frac{E_n^p(\lambda_2)E_n^p(\lambda_3)E_{n'}^{p'}(\lambda_2)E_{n'}^{p'}(\lambda_3)}{\sqrt{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}}\, dS = 0. \tag{1.51}$$

If $n = n'$ and $p = p'$, denote by γ_n^p the normalization constant:

$$\gamma_n^p = \int_{E_{\lambda_1}} \frac{(E_n^p(\lambda_2)E_n^p(\lambda_3))^2}{\sqrt{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}}\, dS. \tag{1.52}$$

1.5.2 Normalization constants γ_n^p

The surface element dS of the ellipsoid E_{λ_1} can be expressed in terms of ellipsoidal coordinates as follows:

$$dS = \frac{(\lambda_2^2-\lambda_3^2)\sqrt{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}}{\sqrt{(k^2-\lambda_2^2)(\lambda_2^2-h^2)(h^2-\lambda_3^2)(k^2-\lambda_3^2)}}\, d\lambda_2\, d\lambda_3 \tag{1.53}$$

In the expression of dS, the quantities $\sqrt{k^2-\lambda_2^2}$ and $\sqrt{h^2-\lambda_3^2}$ are always positive wherever we are in space so that the coefficient in front of $d\lambda_2\, d\lambda_3$ is always positive.

Proposition 1 *The integral of any function of λ_2 and λ_3 over the ellipsoid E_{λ_1} is:*

$$\begin{aligned} \int_{E_{\lambda_1}} & f(\lambda_2,\lambda_3)\, dS = \\ \int_{-h}^{h}\int_{h}^{k} & \left[f_1(\lambda_2,\lambda_3) + f_2(\lambda_2,\lambda_3) + f_3(\lambda_2,\lambda_3) + f_4(\lambda_2,\lambda_3)\right] \\ & \times \frac{(\lambda_2^2-\lambda_3^2)\sqrt{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}}{\sqrt{(k^2-\lambda_2^2)(\lambda_2^2-h^2)(h^2-\lambda_3^2)(k^2-\lambda_3^2)}}\, d\lambda_2\, d\lambda_3 \end{aligned} \tag{1.54}$$

where $f_1(\lambda_2,\lambda_3)$, $f_2(\lambda_2,\lambda_3)$, $f_3(\lambda_2,\lambda_3)$ and $f_4(\lambda_2,\lambda_3)$ are the values of the given function $f(\lambda_2,\lambda_3)$ on the four quarters of the ellipsoid into which it is divided by the planes (XY) and (XZ).

Apply the preceding result to

$$f(\lambda_2,\lambda_3) = \frac{(E_n^p(\lambda_2)E_n^p(\lambda_3))^2}{\sqrt{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}} \tag{1.55}$$

so that

$$\int_{E_{\lambda_1}} f(\lambda_2,\lambda_3)\, dS = \gamma_n^p. \tag{1.56}$$

When we move from one of the four regions $(Y>0, Z>0)$, $(Y>0, Z<0)$, $(Y<0, Z>0)$, $(Y<0, Z<0)$ to another, only the signs of $\sqrt{k^2-\lambda_2^2}$ and $\sqrt{h^2-\lambda_3^2}$ may change in the expression of $E_n^p(\lambda_2)E_n^p(\lambda_3)$. Since $f(\lambda_2,\lambda_3)$ is a function of $(E_n^p(\lambda_2)E_n^p(\lambda_3))^2$, we have: $f_1(\lambda_2,\lambda_3) = f_2(\lambda_2,\lambda_3) = f_3(\lambda_2,\lambda_3) = f_4(\lambda_2,\lambda_3)$ and then:

$$\gamma_n^p = 4 \times \int_{-h}^{h}\int_{h}^{k} \frac{(\lambda_2^2-\lambda_3^2)(E_n^p(\lambda_2)E_n^p(\lambda_3))^2}{\sqrt{(k^2-\lambda_2^2)(\lambda_2^2-h^2)(h^2-\lambda_3^2)(k^2-\lambda_3^2)}} d\lambda_2 d\lambda_3. \quad (1.57)$$

Finally, since $E_n^p(-\lambda_3) = \pm E_n^p(\lambda_3)$ depending on the Lamé's function E_n^p, equation (1.57) can be rewritten as:

$$\gamma_n^p = 8 \times \int_{0}^{h}\int_{h}^{k} \frac{(\lambda_2^2-\lambda_3^2)(E_n^p(\lambda_2)E_n^p(\lambda_3))^2}{\sqrt{(k^2-\lambda_2^2)(\lambda_2^2-h^2)(h^2-\lambda_3^2)(k^2-\lambda_3^2)}} d\lambda_2 d\lambda_3. \quad (1.58)$$

1.5.3 Normalized surface ellipsoidal harmonics

Once the normalization constant is computed, we define a normalized surface ellipsoidal harmonic as:

$$\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3) = \frac{E_n^p(\lambda_2)E_n^p(\lambda_3)}{\sqrt{\gamma_n^p}}, \quad (1.59)$$

and a normalized ellipsoidal harmonic coefficient as:

$$\overline{\alpha_{np}} = \alpha_{np}\sqrt{\gamma_n^p}. \quad (1.60)$$

The solution of Laplace's equation (1.18) can then be rewritten in terms of the normalized surface ellipsoidal harmonics and the normalized ellipsoidal harmonics coefficients:

$$V(\lambda_1,\lambda_2,\lambda_3) = \mu\sum_{n=0}^{\infty}\sum_{p=1}^{2n+1}\overline{\alpha_{np}}\frac{E_n^p(\lambda_1)}{E_n^p(\lambda_1^{ref})}\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3), \quad \lambda_1 \le \lambda_1^{ref}, \quad (1.61)$$

$$V(\lambda_1,\lambda_2,\lambda_3) = \mu\sum_{n=0}^{\infty}\sum_{p=1}^{2n+1}\overline{\alpha_{np}}\frac{F_n^p(\lambda_1)}{F_n^p(\lambda_1^{ref})}\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3), \quad \lambda_1 \ge \lambda_1^{ref}. \quad (1.62)$$

The interest of the normalized surface ellipsoidal harmonics is that the orthogonalization property seen in section 1.5.1 takes the simple form:

$$\int_{E_{\lambda_1}} \frac{\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3)\overline{E_{n'}^{p'}}(\lambda_2)\overline{E_{n'}^{p'}}(\lambda_3)}{\sqrt{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}} dS = \delta_{np}^{n'p'} \quad (1.63)$$

with

$$\delta_{np}^{n'p'} = \begin{cases} 1 & \text{if } n=n' \text{ and } p=p' \\ 0 & \text{else.} \end{cases}$$

This formula is to be used in Chapter 2.

Chapter 2

From spherical harmonics to ellipsoidal harmonics

In this chapter, we want to relate ellipsoidal harmonics to spherical harmonics. Spherical harmonics are widely used because not only are they well understood but we also have the capability to estimate the corresponding coefficients of a gravity field, C_{lm} and S_{lm}, up to high degrees and orders. Our approach is to take advantage of this knowledge to compute the ellipsoidal harmonics, which are much more suitable to model the force environment of irregularly shaped attracting bodies as we will see in Section 2.4. We will first compare these two types of expansion in terms of their mathematical expressions and regions of convergence. Then we will formulate how one can express the ellipsoidal harmonic coefficients, α_{np}, as linear combinations of the spherical harmonics coefficients, C_{lm} and S_{lm}.

In all the following, we consider an attracting body that is best approximated by an ellipsoid $E_{\mathcal{B}}$ of semi-axes $a > b > c > 0$. $E_{\mathcal{B}}$ is called the Brillouin ellipsoid and is defined as the smallest ellipsoid that encloses the body. Similarly, one can define the Brillouin sphere, $S_{\mathcal{B}}$, as the smallest sphere that encloses the body. Generally, the centers of the Brillouin ellipsoid and the Brillouin sphere do not coincide.

2.1 Harmonic expansions of the potential

2.1.1 Spherical harmonics expansion of the potential

One way to express the gravitational potential V as a spherical harmonics expansion is:

$$V(r,\delta,\lambda) = \mu \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{1}{r^{l+1}} P_{lm}(\sin\delta)\,[C_{lm}\cos(m\lambda) + S_{lm}\sin(m\lambda)] \qquad (2.1)$$

where

- $r > 0$, $\delta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\lambda \in [0, 2\pi]$ are the usual spherical coordinates,
- P_{lm} are the associated Legendre's polynomials and have the general expression

$$P_{lm}(\sin\delta) = (\cos\delta)^m \sum_{i=0}^{\lfloor \frac{l-m}{2} \rfloor} \frac{(-1)^i\,(2l-2i)}{2^l\, i!\,(l-i)!\,(l-m-2i)!} (\sin\delta)^{l-m-2i}, \qquad (2.2)$$

- the C_{lm} and S_{lm} have the dimension of a distance to the power l.

2.1.2 Ellipsoidal harmonics expansion of the potential

In section 1.5.3, we expressed the external gravitational potential in terms of normalized ellipsoidal harmonics. If we choose $\lambda_1^{ref} = a$ so that $E_{\lambda_1^{ref}}$ coincides with the Brillouin ellipsoid and rewrite equation (1.62), we have:

$$V(\lambda_1,\lambda_2,\lambda_3) = \mu \sum_{n=0}^{\infty} \sum_{p=1}^{2n+1} \overline{\alpha_{np}}\, \frac{F_n^p(\lambda_1)}{F_n^p(a)} \overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3), \quad \lambda_1 \geq a. \qquad (2.3)$$

2.1.3 Analogies between the two types of expansions

The indices (l, m) and (n, p) are called degree and order of the spherical harmonics expansion, respectively ellipsoidal harmonics expansion. For a given degree $l = n$, there are as many spherical harmonics $(2l + 1)$ as ellipsoidal harmonics $(2n + 1)$. Furthermore, there is a strong analogy in the expressions of the harmonics themselves that we will discuss in the followings.

In section 1.4, we have seen that

$$F_n^p(\lambda_1) = \frac{E_n^p(\lambda_1)}{\lambda_1^{2n+1}}, \quad \text{when } \lambda_1 \text{ is very large.} \qquad (2.4)$$

In addition $E_n^p(\lambda_1) \sim c_0\, \lambda_1^n,\ \lambda_1 \to \infty$, so that

$$F_n^p(\lambda_1) = \frac{c_0}{\lambda_1^{n+1}}, \quad \lambda_1 \to \infty, \tag{2.5}$$

is the analogous of the term $\dfrac{1}{r^{l+1}}$ in the spherical harmonic expansion.
Similarly, one can relate the terms

$$P_{lm}\sin(\delta)\begin{cases}\cos(m\lambda)\\ \sin(m\lambda)\end{cases} \quad \text{and} \quad \overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3) \tag{2.6}$$

to one another since they are functions of the analogous sets of variables (δ, λ) and (λ_2, λ_3) respectively.

The explicit parallelism between both expansions leads us to investigate any relation between the ellipsoidal harmonics coefficients, α_n^p, and the spherical harmonics coefficients, C_{lm} and S_{lm}. This will be the object of Section 2.3.

2.2 Harmonic expansions of the acceleration

When studying the dynamics of a spacecraft, we are more interested in evaluating the acceleration than the potential. In the following, we will derive the acceleration vector $(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z})$ from the expression of the potential V for the two types of expansion.

2.2.1 Spherical harmonics expansion of the acceleration

For the spherical harmonics expansion, the potential V is a function of (r, δ, λ). Thus, according to the chain rule, $\frac{\partial V}{\partial x}$ can be expressed as

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \delta}\frac{\partial \delta}{\partial x} + \frac{\partial V}{\partial \lambda}\frac{\partial \lambda}{\partial x}, \tag{2.7}$$

and similarly for $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$.

The terms $(\frac{\partial V}{\partial r}, \frac{\partial V}{\partial \delta}, \frac{\partial V}{\partial \lambda})$ are obtained by differentiating equation (2.1):

$$\frac{\partial V}{\partial r} = -\mu \sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{(l+1)}{r^{l+2}} P_{lm}(sin\delta)\,[C_{lm}\cos(m\lambda) + S_{lm}\sin(m\lambda)], \tag{2.8}$$

$$\frac{\partial V}{\partial \delta} = \mu \sum_{l=0}^{\infty}\sum_{m=0}^{l} \frac{1}{r^{l+1}} \frac{\partial P_{lm}(\sin\delta)}{\partial \delta}\,[C_{lm}\cos(m\lambda) + S_{lm}\sin(m\lambda)], \tag{2.9}$$

$$\frac{\partial V}{\partial \lambda} = \mu \sum_{l=0}^{\infty} \sum_{m=0}^{l} \frac{1}{r^{l+1}} P_{lm}(\sin\delta)\,[-mC_{lm}\sin(m\lambda) + mS_{lm}\cos(m\lambda)], \quad (2.10)$$

with

$$\frac{\partial P_{lm}(\sin\delta)}{\partial \delta} = -m(\tan\delta)P_{lm}(\sin\delta) + P_{l(m+1)}(\sin\delta). \quad (2.11)$$

One can then relate the spherical acceleration vector $(\frac{\partial V}{\partial r}, \frac{\partial V}{\partial \delta}, \frac{\partial V}{\partial \lambda})$ to the cartesian acceleration vector $(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z})$:

$$\begin{pmatrix} \dfrac{\partial V}{\partial x} \\ \dfrac{\partial V}{\partial y} \\ \dfrac{\partial V}{\partial z} \end{pmatrix} = \begin{pmatrix} \dfrac{x}{r} & \dfrac{-xz}{r^2\sqrt{x^2+y^2}} & \dfrac{-y}{x^2+y^2} \\ \dfrac{y}{r} & \dfrac{-yz}{r^2\sqrt{x^2+y^2}} & \dfrac{x}{x^2+y^2} \\ \dfrac{z}{r} & \dfrac{\sqrt{x^2+y^2}}{r^2} & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial V}{\partial r} \\ \dfrac{\partial V}{\partial \delta} \\ \dfrac{\partial V}{\partial \lambda} \end{pmatrix} \quad (2.12)$$

2.2.2 Ellipsoidal harmonics expansion of the acceleration

Consider the ellipsoidal harmonics expansion of the potential (2.3) and let us first rewrite the Lamé functions of the first kind, $\overline{E_n^p}(\lambda)$, in a more suitable form for the following derivations.

In section 1.3, we defined four types of Lamé functions of the first kind, K, L, M and N. These functions can be rewritten as:

$$\overline{E_n^p}(\lambda) = \psi_n^p(\lambda)\, P_n^p(\lambda), \quad (2.13)$$

with

$$P_n^p(\lambda) = \sum_{i=0}^{m} B_i\,(1 - \frac{\lambda^2}{h^2})^i. \quad (2.14)$$

The leading products $\psi_n^p(\lambda)$ are of the form

$$\lambda^u\, \sqrt{|\lambda^2 - h^2|}^{\,v}\, \sqrt{|\lambda^2 - k^2|}^{\,w} \quad (2.15)$$

with $u, v, w \in \{0, 1\}$. They are determined according to the type of the Lamé function (K, L, M, N) and the parity of n (see Appendix C). The degree m of $P_n^p(\lambda)$ in λ^2 is then equal to $m = \frac{1}{2}\,(n - u - v - w)$.

The ellipsoidal harmonics expansion of the potential then takes the form:

$$V(\lambda_1, \lambda_2, \lambda_3) = \mu \sum_{n=0}^{\infty} \sum_{p=1}^{2n+1} \overline{\alpha_{np}}\, \frac{F_n^p(\lambda_1)}{F_n^p(a)} \psi_n^p(\lambda_2)\psi_n^p(\lambda_3) P_n^p(\lambda_2) P_n^p(\lambda_3), \quad \lambda_1 \geq a. \quad (2.16)$$

The term $\psi_n^p(\lambda_2)\psi_n^p(\lambda_3)$ can be expressed as a function of λ_1 and the cartesian coordinates (x, y, z) (see Appendix C) by using the ellipsoidal-to-cartesian coordinates transformation seen in section 1.1. This is particularly useful when differentiating (2.16) with respect to x, y and z.

For the differentiation of (2.16), we proceed as follows:

$$\Big(\frac{\partial V}{\partial x_i}\Big)_{x_i=x,y,z} = \mu \sum_{n=0}^{\infty} \sum_{p=1}^{2n+1} \frac{\overline{\alpha_{np}}}{F_n^p(a)} \Big[\psi_n^p(\lambda_2)\psi_n^p(\lambda_3) \Big[P_n^p(\lambda_2)P_n^p(\lambda_3) \frac{\partial F_n^p(\lambda_1)}{\partial \lambda_1} \frac{\partial \lambda_1}{\partial x_i} + F_n^p(\lambda_1)P_n^p(\lambda_3) \frac{\partial P_n^p(\lambda_2)}{\lambda_2} \frac{\partial \lambda_2}{\partial x_i} + F_n^p(\lambda_1)P_n^p(\lambda_2) \frac{\partial P_n^p(\lambda_3)}{\lambda_3} \frac{\partial \lambda_3}{\partial x_i} \Big] + F_n^p(\lambda_1)P_n^p(\lambda_2)P_n^p(\lambda_3) \frac{\partial}{\partial x_i} \big(\psi_n^p(\lambda_2)\psi_n^p(\lambda_3)\big) \Big]. \tag{2.17}$$

In the above expressions, most of the terms are perfectly defined, exept for $\frac{\partial F_n^p(\lambda_1)}{\partial \lambda_1}$ and $(\frac{\partial \lambda_1}{\partial x_i}, \frac{\partial \lambda_2}{\partial x_i}, \frac{\partial \lambda_3}{\partial x_i})$ that still need to be determined.

Differentiating equation (1.42), one can show that

$$\frac{\partial F_n^p(\lambda_1)}{\partial \lambda_1} = \frac{1}{E_n^p(\lambda_1)} \Big[F_n^p(\lambda_1) \frac{\partial E_n^p(\lambda_1)}{\partial \lambda_1} - \frac{2n+1}{\sqrt{(\lambda_1^2 - h^2)(\lambda_1^2 - k^2)}} \Big]. \tag{2.18}$$

For the computation of $(\frac{\partial \lambda_1}{\partial x_i}, \frac{\partial \lambda_2}{\partial x_i}, \frac{\partial \lambda_3}{\partial x_i})$, differentiating the ellipsoidal-to-cartesian coordinates equations (1.5), (1.6) and (1.7) with respect to x_i yields a linear system in $(\frac{\partial \lambda_1}{\partial x_i}, \frac{\partial \lambda_2}{\partial x_i}, \frac{\partial \lambda_3}{\partial x_i})$ that can easily been solved. The $(\frac{\partial \lambda_1}{\partial x_i}, \frac{\partial \lambda_2}{\partial x_i}, \frac{\partial \lambda_3}{\partial x_i})$ are given in Appendix C.

2.3 Relation between the α_{np} and the (C_{lm}, S_{lm})

In the following, we want to express the normalized ellipsoidal harmonics coefficients, $\overline{\alpha_{np}}$ (or equivalently the α_{np}), as a function of the spherical harmonics coefficients C_{lm} and S_{lm}. The orthogonalization property seen in Section 1.5.3 will be useful insofar as it will allow us isolate one $\overline{\alpha_{np}}$.

2.3.1 A good use of the orthogonalization property

Applying the orthogonalization property (1.63) to both sides of (2.3) yields:

$$\mu\overline{\alpha_{np}}\frac{F_n^p(\lambda_1)}{F_n^p(a)} = \iint_{E_{\lambda_1}} V(\lambda_1, \lambda_2, \lambda_3)\,\frac{\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3)}{\sqrt{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}}\,dS \tag{2.19}$$

and thus one can isolate $\overline{\alpha_{np}}$ as

$$\overline{\alpha_{np}} = \frac{F_n^p(a)}{F_n^p(\lambda_1)}\iint_{E_{\lambda_1}} \frac{V(\lambda_1, \lambda_2, \lambda_3)}{\mu}\,\frac{\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3)}{\sqrt{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}}\,dS. \tag{2.20}$$

Finally, replacing the potential V by its spherical harmonics expansion (2.1), one can express $\overline{\alpha_{np}}$ as a linear transformation of the C_{lm} and S_{lm}:

$$\overline{\alpha_{np}} = \sum_{l=0}^{\infty}\sum_{m=0}^{l}(A_{np}^{lm}\cdot C_{lm} + B_{np}^{lm}\cdot S_{lm}) \tag{2.21}$$

with A_{np}^{lm} and B_{np}^{lm} coefficients defined by

$$A_{np}^{lm} = \frac{F_n^p(a)}{F_n^p(\lambda_1)}\iint_{E_{\lambda_1}}\frac{dS}{r^{l+1}\sqrt{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}}P_{lm}(\sin\delta)\cos(m\lambda)\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3) \tag{2.22}$$

$$B_{np}^{lm} = \frac{F_n^p(a)}{F_n^p(\lambda_1)}\iint_{E_{\lambda_1}}\frac{dS}{r^{l+1}\sqrt{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}}P_{lm}(\sin\delta)\sin(m\lambda)\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3) \tag{2.23}$$

Some precautions have to be taken for the choice of the ellipsoid E_{λ_1}. This will be developped in section 2.4 *Considerations on convergence.*

2.3.2 Symmetries

Consider the term

$$\frac{dS}{r^{l+1}\sqrt{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}} \tag{2.24}$$

in equations (2.22) and (2.23). We have seen in Section 1.1 that there are eight points on the surface of the ellipsoid E_{λ_1} coresponding to the same $(\lambda_i^2)_{i=1,2,3}$. Let denote these points by $(P_j)_{j=1,\ldots,8}$. According to equations (1.5), (1.6) and (1.7), the P_j's also have identical values for x^2, y^2, and z^2. Therefore:

- they have the same spherical radius r,
- their latitude is $\pm\delta$,

- their longitude is λ, $\pi - \lambda$, $\pi + \lambda$ or $2\pi - \lambda$.

Consequently, the term $\frac{dS}{r^{l+1}\sqrt{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}}$ is identical for all the $(P_j)_{j=1,\cdots,8}$ as long as we choose identical elementary surface area dS.

Consider now the other term,

$$P_{lm}(\sin\delta)\begin{Bmatrix}\cos(m\lambda)\\ \sin(m\lambda)\end{Bmatrix}\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3). \tag{2.25}$$

It has the same norm for each of the $(P_j)_{j=1,\cdots,8}$. By arguing on its sign, one can then deduce some interesting properties for the $\overline{\alpha_{np}}$. Especially, we will see in the following that the coefficients A_{np}^{lm} and B_{np}^{lm} may vanish depending on the parity of n, p, l and m.

We will treat as an example the case where n is even, $1 \leq p \leq (\sigma + 1)$. For all the other cases, we will summarize the results in a table.

2.3.2.1 Example

For n even and $1 \leq p \leq (\sigma+1) = (\frac{n}{2}+1)$, that is $\overline{E_n^p}$ is of type K and thus it is an even polynomial in λ. Therefore, the sign of $\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3)$ is identical for all the $(P_j)_{j=1,\ldots,8}$. We then only need to argue on the sign of $P_{lm}(\sin\delta)\cos(m\lambda)$ and $P_{lm}(\sin\delta)\sin(m\lambda)$. Let us distinguish the four cases:

- l and m even:
  - P_{lm} is an even polynomial in $\sin\delta$.
  - Arguments of symetry for the sign of $\cos(m\lambda)$, $\sin(m\lambda)$ and $P_{lm}(\sin\delta)$:

P_1	δ	λ	$\cos(m\lambda)$	$\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$+$	$+$
P_2	δ	$\pi-\lambda$	$\cos(m\lambda)$	$-\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$+$	$-$
P_3	δ	$\pi+\lambda$	$\cos(m\lambda)$	$\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$+$	$+$
P_4	δ	$2\pi-\lambda$	$\cos(m\lambda)$	$-\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$+$	$-$
P_5	$-\delta$	λ	$\cos(m\lambda)$	$\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$+$	$+$
P_6	$-\delta$	$\pi-\lambda$	$\cos(m\lambda)$	$-\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$+$	$-$
P_7	$-\delta$	$\pi+\lambda$	$\cos(m\lambda)$	$\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$+$	$+$
P_8	$-\delta$	$2\pi-\lambda$	$\cos(m\lambda)$	$-\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$+$	$-$

The last two columns of the table represent the sign of the terms

$$\begin{cases} P_{lm}(\sin\delta)\cos(m\lambda)\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3) \\ P_{lm}(\sin\delta)\sin(m\lambda)\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3) \end{cases}$$

for each of the P_j's. Since both of these terms have respectively constant norms over all the P_j's, the A_{np}^{lm} are a priori non zero whereas the B_{np}^{lm} are zero. More specifically, the A_{np}^{lm} can be rewritten as

$$A_{np}^{lm} = 8\,\frac{F_n^p(a)}{F_n^p(\lambda_1)} \iint_{(E_{\lambda_1})^+} \frac{P_{lm}(\sin\delta)\cos(m\lambda)\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3)}{r^{l+1}\sqrt{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}}\,dS, \quad (2.26)$$

where $(E_{\lambda_1})^+$ denotes the octant of the ellipsoid E_{λ_1} corresponding to $(X>0, Y>0, Z>0)$. This integral is a priori non zero, however it may vanish for some n, p, l and m.

- l and m odd:
  - P_{lm} is an even polynomial in $\sin\delta$.
  - Arguments of symetry for the sign of $\cos(m\lambda)$ and $\sin(m\lambda)$:

P_1	δ	λ	$\cos(m\lambda)$	$\sin(m\lambda)$	$P_{lm}(\sin\delta)$	+	+
P_2	δ	$\pi-\lambda$	$-\cos(m\lambda)$	$\sin(m\lambda)$	$P_{lm}(\sin\delta)$	−	+
P_3	δ	$\pi+\lambda$	$-\cos(m\lambda)$	$-\sin(m\lambda)$	$P_{lm}(\sin\delta)$	−	−
P_4	δ	$2\pi-\lambda$	$\cos(m\lambda)$	$-\sin(m\lambda)$	$P_{lm}(\sin\delta)$	+	−
P_5	$-\delta$	λ	$\cos(m\lambda)$	$\sin(m\lambda)$	$P_{lm}(\sin\delta)$	+	+
P_6	$-\delta$	$\pi-\lambda$	$-\cos(m\lambda)$	$\sin(m\lambda)$	$P_{lm}(\sin\delta)$	−	+
P_7	$-\delta$	$\pi+\lambda$	$-\cos(m\lambda)$	$-\sin(m\lambda)$	$P_{lm}(\sin\delta)$	−	−
P_8	$-\delta$	$2\pi-\lambda$	$\cos(m\lambda)$	$-\sin(m\lambda)$	$P_{lm}(\sin\delta)$	+	−

  Then $A_{np}^{lm} = B_{np}^{lm} = 0$.

- l even and m odd:
  - P_{lm} is an odd polynomial in $\sin\delta$.
  - Arguments of symmetry for the sign of $\cos(m\lambda)$, $\sin(m\lambda)$ and $P_{lm}(\sin\delta)$:

P_1	δ	λ	$\cos(m\lambda)$	$\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$+$	$+$
P_2	δ	$\pi-\lambda$	$-\cos(m\lambda)$	$\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$-$	$+$
P_3	δ	$\pi+\lambda$	$-\cos(m\lambda)$	$-\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$-$	$-$
P_4	δ	$2\pi-\lambda$	$\cos(m\lambda)$	$-\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$+$	$-$
P_5	$-\delta$	λ	$\cos(m\lambda)$	$\sin(m\lambda)$	$-P_{lm}(\sin\delta)$	$-$	$-$
P_6	$-\delta$	$\pi-\lambda$	$-\cos(m\lambda)$	$\sin(m\lambda)$	$-P_{lm}(\sin\delta)$	$+$	$-$
P_7	$-\delta$	$\pi+\lambda$	$-\cos(m\lambda)$	$-\sin(m\lambda)$	$-P_{lm}(\sin\delta)$	$+$	$+$
P_8	$-\delta$	$2\pi-\lambda$	$\cos(m\lambda)$	$-\sin(m\lambda)$	$-P_{lm}(\sin\delta)$	$-$	$+$

Then $A_{np}^{lm} = B_{np}^{lm} = 0$.

- l odd and m even:
  - P_{lm} is an odd polynomial in $\sin\delta$.
  - Arguments of symmetry for the sign of $\cos(m\lambda)$, $\sin(m\lambda)$ and $P_{lm}(\sin\delta)$:

P_1	δ	λ	$\cos(m\lambda)$	$\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$+$	$+$
P_2	δ	$\pi-\lambda$	$\cos(m\lambda)$	$-\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$+$	$-$
P_3	δ	$\pi+\lambda$	$\cos(m\lambda)$	$\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$+$	$+$
P_4	δ	$2\pi-\lambda$	$\cos(m\lambda)$	$-\sin(m\lambda)$	$P_{lm}(\sin\delta)$	$+$	$-$
P_5	$-\delta$	λ	$\cos(m\lambda)$	$\sin(m\lambda)$	$-P_{lm}(\sin\delta)$	$-$	$-$
P_6	$-\delta$	$\pi-\lambda$	$\cos(m\lambda)$	$-\sin(m\lambda)$	$-P_{lm}(\sin\delta)$	$-$	$+$
P_7	$-\delta$	$\pi+\lambda$	$\cos(m\lambda)$	$\sin(m\lambda)$	$-P_{lm}(\sin\delta)$	$-$	$-$
P_8	$-\delta$	$2\pi-\lambda$	$\cos(m\lambda)$	$-\sin(m\lambda)$	$-P_{lm}(\sin\delta)$	$-$	$+$

Then $A_{np}^{lm} = B_{np}^{lm} = 0$.

2.3.2.2 Summary of results

By going through all the cases individually, we obtain the following tables for the A_{np}^{lm} and B_{np}^{lm}. As a reminder, one can relate the value of p to the type of the Lamé function $\overline{E_n^p}$:

- if $1 \leq p \leq (\sigma+1)$, $\overline{E_n^p}$ is of type K,
- if $(\sigma+2) \leq p \leq (n+1)$, $\overline{E_n^p}$ is of type L,
- if $(n+2) \leq p \leq (2n-\sigma+1)$, $\overline{E_n^p}$ is of type M,

- if $(2n-\sigma+2) \leq p \leq (2n+1)$, $\overline{E_n^p}$ is of type N.

A_{np}^{lm}								
type	n even				n odd			
of	l even		l odd		l even		l odd	
$\overline{E_n^p}$	m even	m odd	m even	m odd	m even	m odd	m even	m odd
K	x	0	0	0	0	0	0	x
L	0	0	0	0	0	0	0	0
M	0	x	0	0	0	0	x	0
N	0	0	0	0	0	0	0	0

Table 2.1:

B_{np}^{lm}								
type	n even				n odd			
of	l even		l odd		l even		l odd	
$\overline{E_n^p}$	m even	m odd	m even	m odd	m even	m odd	m even	m odd
K	0	0	0	0	0	0	0	0
L	x	0	0	0	0	0	0	x
M	0	0	0	0	0	0	0	0
N	0	x	0	0	0	0	x	0

Table 2.2:

Using the tables above, the relation between ellipsoidal harmonics coefficients and spherical harmonics coefficients can be simplified. Depending on the values of the degree n and the order p, equation (2.21) reduces to:

- For n even,

– for $1 \leq p \leq \frac{n+2}{2}$,

$$\overline{\alpha_{np}} = \sum_l \sum_m A_{np}^{2l,2m} \cdot C_{2l,2m}, \tag{2.27}$$

– for $\frac{n+4}{2} \leq p \leq (n+1)$,

$$\overline{\alpha_{np}} = \sum_l \sum_m B_{np}^{2l,2m} \cdot S_{2l,2m}, \tag{2.28}$$

– for $(n+2) \leq p \leq \frac{3n+2}{2}$,

$$\overline{\alpha_{np}} = \sum_l \sum_m A_{np}^{2l,2m+1} \cdot C_{2l,2m+1}, \tag{2.29}$$

– for $\frac{3n+4}{2} \leq p \leq (2n+1)$,

$$\overline{\alpha_{np}} = \sum_l \sum_m B_{np}^{2l,2m+1} \cdot S_{2l,2m+1}. \tag{2.30}$$

- For n odd,

– for $1 \leq p \leq \frac{n+1}{2}$,

$$\overline{\alpha_{np}} = \sum_l \sum_m A_{np}^{2l+1,2m+1} \cdot C_{2l+1,2m+1}, \tag{2.31}$$

– for $\frac{n+3}{2} \leq p \leq (n+1)$,

$$\overline{\alpha_{np}} = \sum_l \sum_m B_{np}^{2l+1,2m+1} \cdot S_{2l+1,2m+1}, \tag{2.32}$$

– for $(n+2) \leq p \leq \frac{3n+3}{2}$,

$$\overline{\alpha_{np}} = \sum_l \sum_m A_{np}^{2l+1,2m} \cdot C_{2l+1,2m}, \tag{2.33}$$

– for $\frac{3n+5}{2} \leq p \leq (2n+1)$,

$$\overline{\alpha_{np}} = \sum_l \sum_m B_{np}^{2l+1,2m} \cdot S_{2l+1,2m}. \tag{2.34}$$

2.3.3 The homogeneous triaxial ellipsoid

Consider now the special case where the attracting body is a homogeneous triaxial ellipsoid. Balmino (see [2]) states that the spherical harmonics expansion of the gravitational potential involves only the $C_{2l,2m}$. Then, according to the result (2.21), one have:

$$\overline{\alpha_{np}} = \sum_l \sum_m A_{np}^{2l,2m} \cdot C_{2l,2m}. \tag{2.35}$$

In section (2.3.2.2), we have seen that:

$$A_{np}^{2l,2m} = 0 \quad \begin{cases} \text{when } n \text{ is even and } (\frac{n}{2}+2) \leq p \leq (2n+1), \\ \text{when } n \text{ is odd.} \end{cases} \tag{2.36}$$

Thus, $\overline{\alpha_n^p}$ is zero for the same n and p. In the ellispoidal harmonics expansion of the potential, a lot of terms are removed and it remains:

$$\begin{aligned} V = \mu \; \Big(& \overline{\alpha_{01}} \frac{F_0^1(\lambda_1)}{F_0^1(a)} \overline{E_0^1}(\lambda_2) \overline{E_0^1}(\lambda_3) \\ & + \overline{\alpha_{21}} \frac{F_2^1(\lambda_1)}{F_2^1(a)} \overline{E_2^1}(\lambda_2) \overline{E_2^1}(\lambda_3) + \overline{\alpha_{22}} \frac{F_2^2(\lambda_1)}{F_2^2(a)} \overline{E_2^2}(\lambda_2) \overline{E_2^2}(\lambda_3) \\ & + \big(\text{terms of degree } 4, 6, 8, \ldots\big) \Big) \end{aligned} \tag{2.37}$$

A similar result has already been proved by Garmier (see [8]), the approach however is different. Garmier uses Ivory's Method (see [11]) and shows that we only need three coefficients, α_{01}, α_{21} and α_{22}, to represent the potential of an homogeneous triaxial ellipsoid.

2.4 Convergence considerations

The use of spherical harmonics is well suited to planets with a high degree of spherical symmetry. Indeed spherical harmonics expansions of the potential and the acceleration are convergent outside the Brillouin sphere. Inside the Brillouin sphere however, severe divergence may occur. Moreover, the deeper is the evaluation point inside the Brillouin sphere, the stronger is the divergence. For Earth-like bodies, the Brillouin sphere has a good match with the surface of the body and thus spherical harmonics are a good choice. If now, we consider bodies such as asteroids or comets, the volume between the surface of the body and the Brillouin sphere, where divergences of the spherical harmonics expansion may occur, can be very large. This may be a serious problem when evaluating the potential and the acceleration at close range. For such bodies, ellipsoidal coordinates are much more appropriate to describe the surface of the body. Solving Laplace's equation in terms of ellipsoidal coordinates gives rise to ellipsoidal harmonics expansion for the potential and the acceleration which are convergent in all space external to the Brillouin ellipsoid. This type of expansion has the advantage of considerably reducing the region of possible divergence since an irregularly shaped body is better fit by an ellipsoid than by a sphere (see Figure 2.1).

In section 2.3.1, the linear transformation,

$$\overline{\alpha_{np}} = f(C_{lm}, S_{lm}), \tag{2.38}$$

has been established by replacing the potential V by its spherical harmonic expansion into

$$\overline{\alpha_{np}} = \frac{F_n^p(a)}{F_n^p(\lambda_1)} \iint_{E_{\lambda_1}} \frac{V(\lambda_1, \lambda_2, \lambda_3)}{\mu} \frac{\overline{E_n^p}(\lambda_2)\overline{E_n^p}(\lambda_3)}{\sqrt{(\lambda_1^2 - \lambda_2^2)(\lambda_1^2 - \lambda_3^2)}} \, dS. \tag{2.39}$$

Because the spherical harmonics expansion is convergent outside the Brillouin sphere and can exhibit severe divergences elsewhere, we have to take some pre-

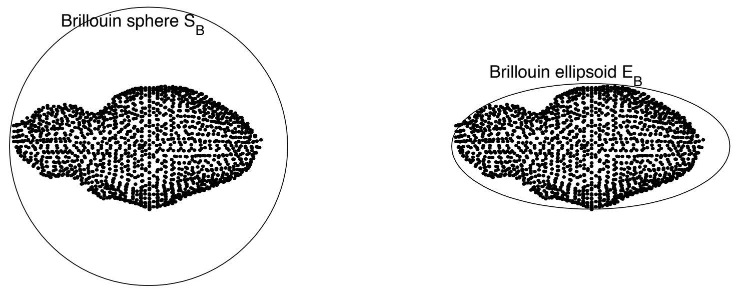


Figure 2.1: Brillouin sphere $S_{\mathcal{B}}$ and Brillouin ellipsoid $E_{\mathcal{B}}$

cautions for the choice of E_{λ_1} on which the potential V needs to be evaluated. The ellipsoid E_{λ_1} has to be chosen such that it encloses the Brillouin sphere, that is its semi-minor axis must be greater than the radius $r_{\mathcal{B}}$ of the Brillouin sphere:

$$\lambda_1^2 - k^2 \geq r_{\mathcal{B}}^2 \tag{2.40}$$

Chapter 3

Numerical methods

Ellipsoidal harmonics are much more appropriate than spherical harmonics to represent the gravity of irregularly shaped attracting bodies. However, their expression is much more complex. Spherical harmonics are defined in terms of the associated Legendre polynomials that are known for any degree and order. On the contrary, ellipsoidal coordinates and Lamé functions can not be as easily handled. In this chapter, we will develop some useful numerical methods

- to compute the ellipsoidal coordinates of any point in space,
- to express Lamé functions of the first kind for high degrees and orders and compute the corresponding normalization constants,
- finally, to compute Lamé functions of the second kind.

In the second part of this chapter, we will focus on the numerical evaluation of the ellipsoidal harmonics coefficients, α_{np}.

3.1 Computation of the ellipsoidal coordinates

The ellipsoidal coordinates $(\lambda_1, \lambda_2, \lambda_3)$ are originally defined as solutions of

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\lambda^2 - h^2} + \frac{z^2}{\lambda^2 - k^2} = 1, \tag{3.1}$$

which can be rewritten as a third order equation in λ^2

$$\lambda^6 + a_1\,\lambda^4 + a_2\,\lambda^2 + a_3 = 0, \tag{3.2}$$

with

$$\begin{cases} a_1 = -(x^2 + y^2 + z^2 + h^2 + k^2), \\ a_2 = (h^2 + k^2)\, x^2 + k^2\, y^2 + h^2\, z^2 + h^2\, k^2, \\ a_3 = -h^2\, k^2\, x^2. \end{cases} \tag{3.3}$$

Explicit solutions of (3.2) are given in [8]:

$$\begin{cases} \lambda_1^2 = 2\sqrt{Q}\, \cos(\frac{\theta}{3}) - \frac{a_1}{3}, \\ \lambda_2^2 = 2\sqrt{Q}\, \cos(\frac{\theta}{3} + \frac{4\pi}{3}) - \frac{a_1}{3}, \\ \lambda_2^2 = 2\sqrt{Q}\, \cos(\frac{\theta}{3} + \frac{2\pi}{3}) - \frac{a_1}{3}, \end{cases} \tag{3.4}$$

with

$$Q = \frac{a_1^2 - 3a_2}{9}, \qquad R = \frac{9a_1\, a_2 - 27a_3 - 2a_1^3}{54}, \qquad \cos\theta = \frac{R}{\sqrt{Q^3}}.$$

Instead of using directly these formulas to compute $(\lambda_1, \lambda_2, \lambda_3)$ in terms of (x, y, z), we will rather solve equation (3.1) numerically using a false position method (see [13]). Given x_1 and x_2, the false position method returns the root of $f(x) = 0$ known to lie in $[x_1, x_2]$. In our case, we choose our initial guesses as follows:

$$\begin{cases} x_1 = \lambda^2 - \epsilon, \\ x_2 = \lambda^2 + \epsilon, \end{cases} \tag{3.5}$$

where λ_i^2 $(i = 1, 2, 3)$ is given by (3.4). ϵ has to be carefully chosen so that $f(x_1) \cdot f(x_2) < 0$ and it is small enough to warranty a rapid convergence to the desired root. From the λ_i^2 $(i = 1, 2, 3)$, $(\lambda_1, \lambda_2, \lambda_3)$ are determined according to the signs of (x, y, z) as mentioned in Chapter 1.

3.2 Computation of the Lamé functions of the first kind

In this section, we focus on the numerical computation of Lamé functions for high degrees and orders. Explicit expressions for the $E_n^p(x)$ exist up to $n = 3$ (see Appendix A). For $n > 3$, we will use the work of S. Ritter and H.-J. Dobner. (see [14] and [6]).

3.2.1 Unnormalized Lamé functions of the first kind

In section 2.2.2, Lamé functions have been rewritten in the general form:

$$E_n^p(\lambda) = \psi_n^p(\lambda)\, P_n^p(\lambda), \tag{3.6}$$

with

$$P_n^p(\lambda) = \sum_{i=0}^{m} B_i \, (1 - \frac{\lambda^2}{h^2})^i. \tag{3.7}$$

The leading products $\psi_n^p(\lambda)$ are well known and are listed in Appendix C. We then only need to compute the coefficients B_i, $i = 0, \ldots, m$.

If we substitute (3.6) into Lamé equation (1.30), we obtain a three-term recurrence relation for the B_i's,

$$a_i \, B_{i-1} + (b_i - K) \, B_i + c_i \, B_{i+1} = 0, \tag{3.8}$$

for $i = 0, \ldots, m$ with $B_{-1} = B_{m+1} = 0$. The parameter K is the same as the one in Lamé equation (1.30). Concerning the other separation parameter H, the substitution yields $H = n(n+1)$.
The problem of the determination of the $B_i's$ is then reduced to the eigenvalue problem

$$\mathbf{A} \, \mathbf{V} = K \, \mathbf{V}, \tag{3.9}$$

$$\text{with} \quad \mathbf{A} = \begin{pmatrix} b_0 & c_0 & 0 & \cdots & 0 \\ a_1 & b_1 & c_1 & \ddots & \vdots \\ 0 & a_2 & b_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c_{m-1} \\ 0 & \cdots & 0 & a_m & b_m \end{pmatrix} \quad \text{and} \quad \mathbf{V} = \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ \vdots \\ B_m \end{pmatrix}.$$

The coefficients a_i, b_i, c_i depend on the parity of n and on the type of the Lamé function. They are listed in Appendix D.

The matrix $\mathbf{A}$ is similar to a symmetric tridiagonal matrix $\mathbf{A}'$. Let $\mathbf{S} = diag(s_k)$, $k = 0, 1, \ldots, m$, with:

$$\begin{cases} s_0 = 1 \\ s_{k+1} = \sqrt{\dfrac{c_k}{a_{k+1}}} \, s_k \quad \text{for} \quad k = 0, \ldots, m-1. \end{cases} \tag{3.10}$$

Then $\mathbf{A}' = \mathbf{S}\,\mathbf{A}\,\mathbf{S^{-1}}$ yields

$$\mathbf{A}' = \begin{pmatrix} b_0 & d_1 & 0 & \cdots & 0 \\ d_1 & b_1 & d_2 & \ddots & \vdots \\ 0 & d_2 & b_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & d_m \\ 0 & \cdots & 0 & d_m & b_m \end{pmatrix} \tag{3.11}$$

with

$$d_k = a_k \,\frac{s_k}{s_{k-1}} = c_{k-1}\,\frac{s_{k-1}}{s_k}, \quad k = 1, \ldots, m. \tag{3.12}$$

The eigenvalues of $\mathbf{A}$ and $\mathbf{A}'$ are identical. If K and $\mathbf{W}$ repectively define an eigenvalue and an eigenvector of $\mathbf{A}'$, then

$$\mathbf{A}'\,\mathbf{W} = K\,\mathbf{W} \qquad \Leftrightarrow \qquad \mathbf{A}\,(\mathbf{S^{-1}}\,\mathbf{W}) = K\,(\mathbf{S^{-1}}\,\mathbf{W}). \tag{3.13}$$

To determine the eigenvalues K and eigenvectors $\mathbf{W}$ of $\mathbf{A}'$, we use a QR-algorithm. The corresponding eigenvectors V of $\mathbf{A}$ are then obtained by:

$$\mathbf{V} = \mathbf{S^{-1}}\,\mathbf{W}. \tag{3.14}$$

To have a unique set of B_i's, we impose $B_m = 1$.

3.2.2 Computation of the normalization constants γ_n^p

Normalized surface ellipsoidal harmonics have been defined in section 1.5 and used in Chapter 2 to relate the α_{np} to the C_{lm} and S_{lm}. The numerical method above returns unnormalized Lamé functions of the first kind. In the following, we describe a numerical method to compute the normalization constants γ_n^p associated with the "unnormalized" Lamé functions of the first kind.

We have seen in section 1.5.2 that

$$\gamma_n^p = 8 \times J \tag{3.15}$$

with

$$J = \int_0^h \int_h^k \frac{(\lambda_2^2 - \lambda_3^2)(E_n^p(\lambda_2)E_n^p(\lambda_3))^2}{\sqrt{(k^2-\lambda_2^2)(\lambda_2^2-h^2)(h^2-\lambda_3^2)(k^2-\lambda_3^2)}} d\lambda_2 d\lambda_3. \tag{3.16}$$

Define

$$dS_2 = \frac{d\lambda_2}{\sqrt{(\lambda_2^2 - h^2)(k^2 - \lambda_2^2)}}, \qquad dS_3 = \frac{d\lambda_3}{\sqrt{(h^2 - \lambda_3^2)(k^2 - \lambda_3^2)}},$$

and

$$I_1 = \int_h^k (E_n^p(\lambda_2))^2 \, dS_2, \qquad I_2 = \int_h^k (E_n^p(\lambda_2))^2 \, \lambda_2^2 \, dS_2,$$
$$I_3 = \int_0^h (E_n^p(\lambda_3))^2 \, dS_3, \qquad I_4 = \int_0^h (E_n^p(\lambda_3))^2 \, \lambda_3^2 \, dS_3.$$

J can then be rewritten as

$$J = I_2 \, I_3 - I_1 \, I_4. \tag{3.17}$$

Expressing I_1, I_2, I_3 and I_4 in terms of 2 basic elliptic integrals:

Consider the following change of variable

$$\Lambda = 1 - \frac{\lambda^2}{h^2}, \; \lambda \in \{\lambda_2, \lambda_3\}. \tag{3.18}$$

We have seen in section 2.2.2 that the $E_n^p(\lambda)$ are of the form:

$$E_n^p(\lambda) = \psi_n^p(\lambda) \sum_{i=0}^{m} B_i \, \Lambda^i, \tag{3.19}$$

$$\text{with} \quad \begin{cases} \psi_n^p(\lambda) = \lambda^u \sqrt{|\lambda^2 - h^2|}^{\,v} \sqrt{|\lambda^2 - k^2|}^{\,w}, \; (u, v, w) \in \{0, 1\}, \\ m = \frac{1}{2}(n - u - v - w). \end{cases}$$

Since

$$\Big(\sum_{i=0}^{m} B_i \, \Lambda^i \Big)^2 = \sum_{j=0}^{2m} D_j \, \Lambda^j, \tag{3.20}$$

$$\text{with} \quad D_j = \sum_{k=max(j-m,0)}^{min(j,m)} B_k \, B_{j-k}, \tag{3.21}$$

the $(E_n^p(\lambda))^2$ can be rewritten as a polynomial in Λ:

$$\Big(E_n^p(\lambda) \Big)^2 = \sum_{j=0}^{n} C_j \, \Lambda^j \tag{3.22}$$

The expression of the C_j's in terms of the D_j's is detailed in Appendix E. The integrals I_1, I_2, I_3 and I_4 are then elliptic integrals.

Now define

$$d\Sigma_2 = \frac{d\Lambda_2}{\sqrt{1-\Lambda_2}\,\sqrt{h^2\,\Lambda_2 + k^2 - h^2}\,\sqrt{-\Lambda_2}} \tag{3.23}$$

and

$$d\Sigma_3 = \frac{d\Lambda_3}{\sqrt{1-\Lambda_3}\,\sqrt{h^2\,\Lambda_3 + k^2 - h^2}\,\sqrt{\Lambda_3}}. \tag{3.24}$$

A change of variable from λ to Λ in the expressions of dS_2 and dS_3 yields

$$dS_2 = -\left(\frac{1}{2}\right) d\Sigma_2, \tag{3.25}$$

$$dS_3 = -\left(\frac{1}{2}\right) d\Sigma_3, \tag{3.26}$$

and substituting (3.22), (3.25) and (3.26) into the expression of I_1, I_2, I_3 and I_4, we obtain:

$$\left.\begin{matrix} I_1 \\ I_3 \end{matrix}\right\} = \sum_{j=0}^{n} -\left(\frac{1}{2}\right) C_j\,K_j \;=\; \sum_{j=0}^{n} \Gamma_j\,K_j \tag{3.27}$$

$$\begin{aligned}\left.\begin{matrix} I_2 \\ I_4 \end{matrix}\right\} &= -\left(\frac{h^2}{2}\right) C_0\,K_0 + \sum_{j=1}^{n} -\left(\frac{h^2}{2}\right)(C_j - C_{j-1})\,K_j + \left(\frac{h^2}{2}\right) C_n\,K_{n+1} \\ &= \sum_{j=0}^{n+1} \tilde{\Gamma}_j\,K_j \end{aligned} \tag{3.28}$$

where

$$K_j(\lambda_i) = \int_a^b \frac{\Lambda_i^j}{\sqrt{1-\Lambda_i}\,\sqrt{h^2\,\Lambda_i + k^2 - h^2}\,\sqrt{\mp\Lambda_i}}\,d\Lambda_i\,, \qquad i \in \{2,3\}. \tag{3.29}$$

For I_1 and I_2, $a = 1 - \dfrac{k^2}{h^2}$, $b = 0$ and we take $\sqrt{-\Lambda}$ in the expression of K_j. For I_3 and I_4, $a = 0$, $b = 1$ and we take $\sqrt{\Lambda}$ in the expression of K_j.

By differentiating $\Lambda^j\,\sqrt{1-\Lambda}\sqrt{h^2\Lambda + k^2 - h^2}\sqrt{\mp\Lambda}$ and reintegrating between a and b, we come up with the following recurrence relation on the K_j:

$$(j + \frac{3}{2})\,h^2\,K_{j+2} - (j+1)\,(2h^2 - k^2)\,K_{j+1} - (j + \frac{1}{2})\,(k^2 - h^2)\,K_j = 0. \tag{3.30}$$

Using this recurrence formula, one can expressed the integrals I_1, I_2, I_3 and I_4 in terms of K_0 and K_1:

$$I_1 = \alpha\,K_0(\lambda_2) + \beta\,K_1(\lambda_2), \tag{3.31}$$

$$I_3 = \alpha\, K_0(\lambda_3) + \beta\, K_1(\lambda_3), \tag{3.32}$$

$$I_2 = A\, K_0(\lambda_2) + B\, K_1(\lambda_2), \tag{3.33}$$

$$I_4 = A\, K_0(\lambda_3) + B\, K_1(\lambda_3). \tag{3.34}$$

Clenshaw's Recurrence Algorithm:

To determine the coefficients α, β, A and B, we use the Clenshaw's Recurrence Formula referenced in [13]. The recurrence relation (3.30) can be rewritten as:

$$K_{j+1} = \alpha(j)\, K_j + \beta(j)\, K_{j-1} \tag{3.35}$$

$$\text{with} \quad \alpha(j) = \frac{2j}{2j+1}\left(2 - \frac{k^2}{h^2}\right), \quad \beta(j) = \frac{2j-1}{2j+1}\left(\frac{k^2}{h^2} - 1\right).$$

Define the quantities y_j $(j = n, n-1, \ldots, 0)$ and $\tilde{y}_j$ $(j = n+1, n, \ldots, 0)$ by:

$$\begin{cases} y_{n+2} = y_{n+1} = 0, \\ y_j = \alpha(j)\, y_{j+1} + \beta(j+1)\, y_{j+2} + \Gamma_j, \qquad j = n, n-1, \ldots, 0. \end{cases} \tag{3.36}$$

$$\begin{cases} \tilde{y}_{n+3} = \tilde{y}_{n+2} = 0, \\ \tilde{y}_j = \alpha(j)\, \tilde{y}_{j+1} + \beta(j+1)\, \tilde{y}_{j+2} + \tilde{\Gamma}_j, \qquad j = n+1, n, \ldots, 0. \end{cases} \tag{3.37}$$

Clenshaw's recurrence algorithm yields:

$$\left.\begin{matrix} I_1 \\ I_3 \end{matrix}\right\} = (y_0 - \alpha(0)\, y_1)\, K_0 + y_1\, K_1, \tag{3.38}$$

$$\left.\begin{matrix} I_2 \\ I_4 \end{matrix}\right\} = (\tilde{y}_0 - \alpha(0)\, \tilde{y}_1)\, K_0 + \tilde{y}_1\, K_1. \tag{3.39}$$

Then $J = I_2\, I_3 - I_1\, I_4$ can be rewritten as

$$J = (A\beta - \alpha B)\left(K_0(\lambda_2)\, K_1(\lambda_3) - K_1(\lambda_2)\, K_0(\lambda_3)\right) \tag{3.40}$$

$$= (\tilde{y}_0\, y_1 - y_0\, \tilde{y}_1)\left(K_0(\lambda_2)\, K_1(\lambda_3) - K_1(\lambda_2)\, K_0(\lambda_3)\right) \tag{3.41}$$

Finally $\left(K_0(\lambda_2)\, K_1(\lambda_3) - K_1(\lambda_2)\, K_0(\lambda_3)\right)$ can be rewritten in terms of a basic elliptic integral:

$$K_0(\lambda_2)\, K_1(\lambda_3) - K_1(\lambda_2)\, K_0(\lambda_3) = \frac{4}{h^2} \int_0^h \int_h^k (\lambda_2^2 - \lambda_3^2)\, dS_2\, dS_3$$

$$= \frac{2\,\pi}{h^2}, \tag{3.42}$$

so that

$$J = (\tilde{y}_0\, y_1 - y_0\, \tilde{y}_1)\, \frac{2\,\pi}{h^2}. \tag{3.43}$$

3.3 Computation of the Lamé functions of the second kind

Lamé functions of the second kind have been defined in section 1.4 by

$$F_n^p(\lambda_1) = (2n+1)\, E_n^p(\lambda_1) \int_{\lambda_1}^{\infty} \frac{du}{\left(E_n^p(u)\right)^2 \sqrt{(u^2-k^2)(u^2-h^2)}}. \tag{3.44}$$

Denote by $I_n^p(\lambda_1)$,

$$\int_{\lambda_1}^{\infty} \frac{du}{\left(E_n^p(u)\right)^2 \sqrt{(u^2-k^2)(u^2-h^2)}}. \tag{3.45}$$

The evaluation of $I_n^p(\lambda_1)$ will be performed by a numerical integration. $I_n^p(\lambda_1)$ can be rewritten as an integral with finite bounds. By changing the variable u to $s = \frac{1}{u}$, one have:

$$I_n^p(\lambda_1) = \int_0^{1/\lambda_1} \frac{ds}{\left(E_n^p(\frac{1}{s})\right)^2 \sqrt{(1-k^2 s^2)(1-h^2 s^2)}} \tag{3.46}$$

The integrand is smooth, in the sense of being "well-approximated by a polynomial" over the interval $[0, 1/\lambda_1]$. Thus a Gauss-Legendre quadrature is very suitable to obtain an accurate estimate of $I_n^p(\lambda_1)$. The Gauss-Legendre quadrature routine calculates a set of absissas x_j and weights w_j such that

$$I_n^p(\lambda_1) \approx \sum_{j=1}^{N} w_j \frac{1}{\left(E_n^p(\frac{1}{x_j})\right)^2 \sqrt{(1-k^2 x_j^2)(1-h^2 x_j^2)}}. \tag{3.47}$$

A numerical method for the evaluation of the E_n^p has been given in section 3.2.1. The F_n^p are then easily accessible from equations (3.44) and (3.47).

3.4 Numerical evaluation of the α_{np}

We have seen in Chapter 2 that the ellipsoidal harmonic coefficients α_{np} can be expressed as a linear combination of the spherical harmonic coefficients C_{lm}

and S_{lm}:

$$\overline{\alpha_{np}} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} (A_{np}^{lm} \cdot C_{lm} + B_{np}^{lm} \cdot S_{lm}). \tag{3.48}$$

The expressions of A_{np}^{lm} and B_{np}^{lm} involve an integral over the surface of an ellipsoid that vanish in many cases because of symmetries. However, when a priori non zero, the integrals can not be computed analyticaly. The main problem that arises in its analytical evaluation is that it involves both types of coordinates, spherical and ellipsoidal, with no simple relations between them.

Once more, we switch to a numerical integration method. The numerical evaluation of the A_{np}^{lm} and B_{np}^{lm} can be performed by "tessellating" the surface of the ellipsoid into a number of small regions (see Figure 3.1). Each region is represented by a flat triangular plate with its vertices on the surface of the ellipsoid. To first approximation, we can treat the integrand as being a constant over each plate. The integral can then be found by summing up the contributions of each plate over the surface of the ellipsoid.

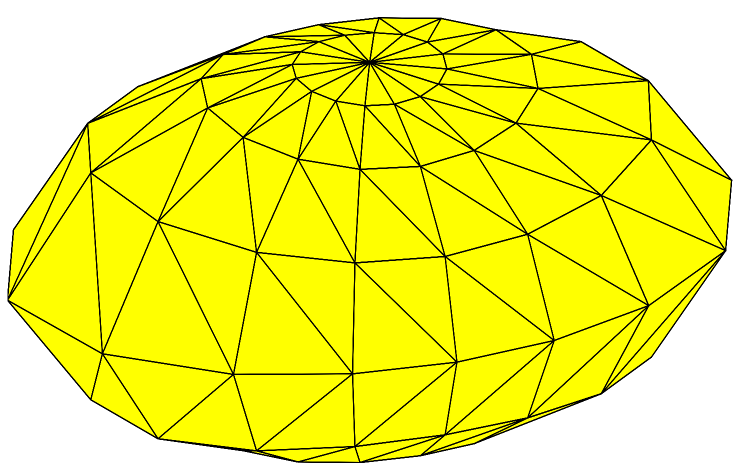


Figure 3.1: "Tessellated surface" of the ellipsoid E_{λ_1}

Chapter 4

Numerical results

In this chapter, we focus on some numerical computations of the ellipsoidal harmonics expansion. Our objective is to validate the theoretical results of the previous chapters. The numerical methods described in Chapter 3 have been incorporated into C code and simulations have been conducted for two asteroids, Asteroid 4179 Toutatis [12] and Asteroid 433 Eros [17].

4.1 Simulation setup

In the following, we evaluate some ellipsoidal harmonics expansions for the external space of an attracting body. Based on the method developed in Chapter 2, we need to know the shape model of the body and its spherical harmonics coefficients.

From the shape model of the body, one can determine the Brillouin ellipsoid $E_{\mathcal{B}}$ (smallest ellipsoid enclosing the attracting body). The Brillouin ellipsoid not only delimits the region in space where the ellipsoidal harmonics expansion converges but also defines the system of ellipsoidal coordinates $(\lambda_1, \lambda_2, \lambda_3)$.
The evaluation of the ellipsoidal harmonics coefficients is subject to the knowledge of the C_{lm}, S_{lm}, A^{lm}_{np} and B^{lm}_{np}. The spherical harmonics coefficients C_{lm} and S_{lm} are given; however, the A^{lm}_{np} and B^{lm}_{np} will be evaluated by performing a numerical integration over the surface of an ellipsoid E_{λ_1} as described in section 3.4. The choice of the ellipsoid E_{λ_1} is subject to the constraints that it must be

confocal to the Brillouin ellipsoid $E_{\mathcal{B}}$ and enclose the Brillouin sphere $S_{\mathcal{B}}$ (see section 2.4). In order to minimize the surface of integration we choose for E_{λ_1} the smallest ellipsoid that encloses $S_{\mathcal{B}}$. It is characterized by the following set of semi-axes:

$$\begin{cases} a = \sqrt{k^2 + r_{\mathcal{B}}^2} \\ b = \sqrt{k^2 - h^2 + r_{\mathcal{B}}^2} \\ c = r_{\mathcal{B}} \end{cases}, \tag{4.1}$$

where $r_{\mathcal{B}}$ is the radius of the Brillouin sphere $S_{\mathcal{B}}$.
Now, when "tessellating" the surface of E_{λ_1}, one can adjust with the increment in latitude $\Delta\delta$ and the increment in longitude $\Delta\lambda$. The smaller these increments are set, the larger is the number of faces and therefore the better is the match between the "tessellated" surface and the true surface of E_{λ_1}.

4.2 Numerical results for the potential

We want to evaluate the ellipsoidal harmonics expansion of the potential V_e for various degrees and compare it with a reference potential V_{ref}. As long as we perform these evaluations of the potential outside the Brillouin sphere $S_{\mathcal{B}}$, it is convenient to choose a spherical harmonics expansion of the potential as our reference. Define the error ϵ as:

$$\epsilon = \frac{|V_e - V_{ref}|}{V_{ref}}. \tag{4.2}$$

It is interesting to study the effect of the level of discretization of the "tessellated surface" E_{λ_1} on the error ϵ. The accuracy of the ellipsoidal harmonics coefficients α_{np} increases as the number of faces of the "tessellated surface" gets larger. We will consider three models:

- model with 1520 faces corresponding to latitude and longitude increments $\Delta\delta = \Delta\lambda = 9\,\text{deg}$,
- model with 6240 faces corresponding to $\Delta\delta = \Delta\lambda = 4.5\,\text{deg}$,
- model 14160 faces corresponding to $\Delta\delta = \Delta\lambda = 3\,\text{deg}$.

4.2.1 External potential of Eros

For Asteroid 433 Eros, the Brillouin ellipsoid is characterized by:

$$\begin{cases} a = 17.556 \text{ km} \\ b = 8.633 \text{ km} \\ c = 6.074 \text{ km} \end{cases} . \tag{4.3}$$

The Brillouin sphere has then radius $r_{\mathcal{B}} = 17.556$ km and the ellipsoid E_{λ_1} has semi-axes:

$$\begin{cases} a = 24.074 \text{ km} \\ b = 18.597 \text{ km} \\ c = 17.556 \text{ km} \end{cases} . \tag{4.4}$$

The reference potential V_{ref} is a spherical harmonics expansion of degree and order 12.

Evaluation of the error ϵ on the ellipsoid E_{λ_1}

Annex F represents the error ϵ evaluated on the ellipsoid E_{λ_1} as a function of latitude and longitude for various degrees of the ellipsoidal harmonics expansion. The α_{np} were computed using the 14160 faces model for the "tessellated surface" E_{λ_1}.

Let us define two interesting types of error ϵ: a "maximum error" and a "weighted error". The former is simply defined by the max(ϵ) over the surface of the ellipsoid E_{λ_1}. The later is the error ϵ weighted on the "tessellated surface" area. If ϵ_i denotes the error ϵ evaluated on the face i with area dS_i of the "tessellated surface" E_{λ_1}, one define the "weighted error" as:

$$\text{"weighted error"} = \frac{\sum_i \epsilon_i \, dS_i}{\sum_i dS_i}. \tag{4.5}$$

Figures 4.1 and 4.2 represent respectively the "maximum error" and the "weighted error" as a function of the degree of the ellipsoidal harmonics expansion. For each case, three curves are shown corresponding to the three "tessellated surface" models.

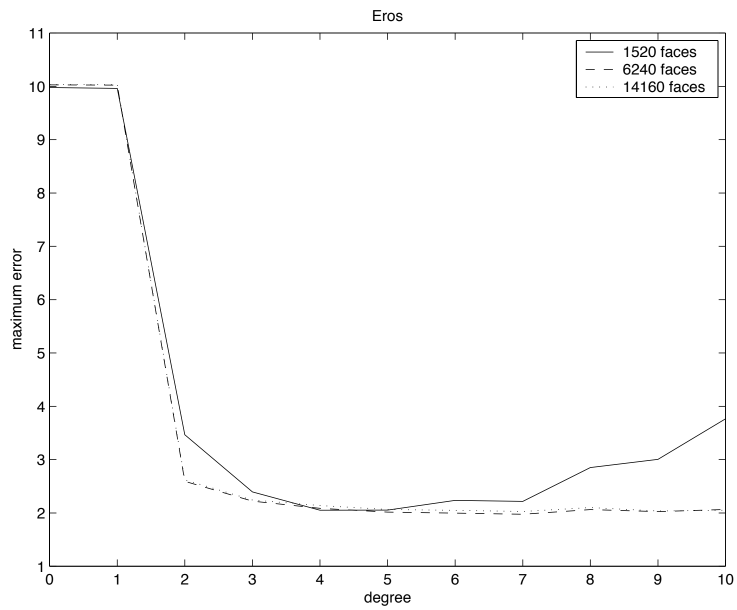


Figure 4.1: Maximum error for Eros

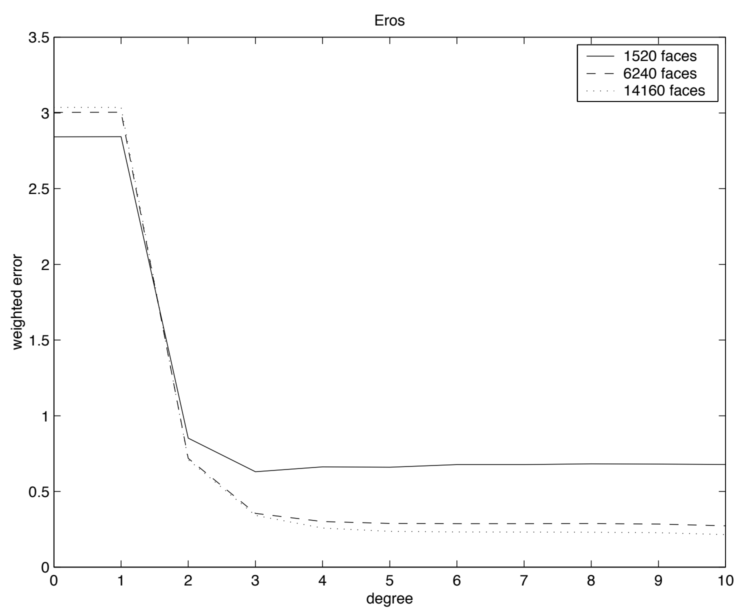


Figure 4.2: Weighted error for Eros

4.2.2 External potential of Toutatis

For Asteroid 4179 Toutatis, the Brillouin ellipsoid is characterized by:

$$\begin{cases} a = 2.517 \text{ km} \\ b = 1.174 \text{ km} \\ c = 0.980 \text{ km} \end{cases} . \tag{4.6}$$

The Brillouin sphere has then radius $r_{\mathcal{B}} = 2.517$ km and the ellipsoid E_{λ_1} has semi-axes:

$$\begin{cases} a = 3.421 \text{ km} \\ b = 2.598 \text{ km} \\ c = 2.517 \text{ km} \end{cases} . \tag{4.7}$$

The reference potential V_{ref} is a spherical harmonics expansion of degree and order 16.

Evaluation of the error ϵ on the ellipsoid E_{λ_1}

Figures 4.3 and 4.4 represent respectively the "maximum error" and the "weighted error" as a function of the degree of the ellipsoidal harmonics expansion. For each case, three curves are shown corresponding to the three "tessellated surface" models.

4.3 Interpretation

The overall observation for Figures 4.1, 4.2, 4.3, 4.4 is that the errors decrease as we add higher order terms to the ellipsoidal harmonics expansion of the potential. Now looking more precisely at the plots, we observe

- that the "maximum error" for the 1520 faces model first decreases, then increases as the degree of the ellipsoidal expansion gets larger both for Eros and Toutatis;
- also that the errors never really go to zero and seem to stabilize.

The behavior of the "maximum error" for the model with 1520 faces is interesting insofar as it does not show up as obviously for the two other models. This indeed reveals that the level of discretization of the "tessellated surface" E_{λ_1}

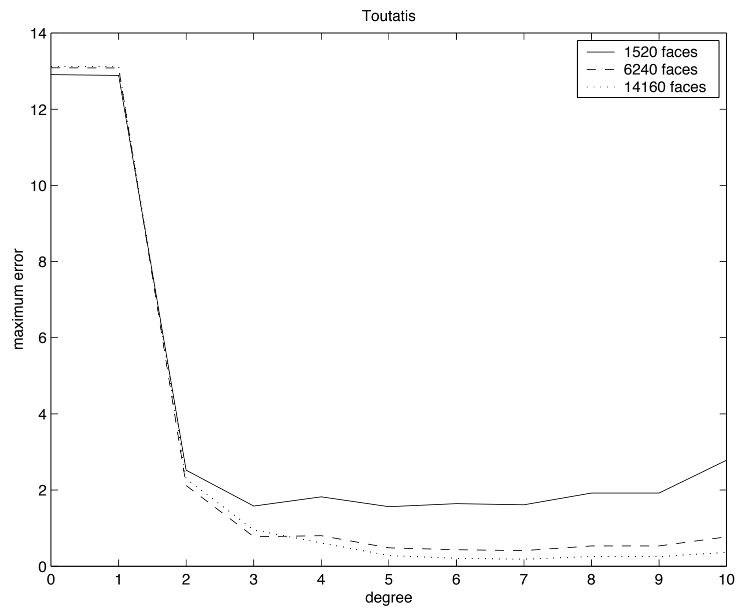


Figure 4.3: Maximum error for Toutatis

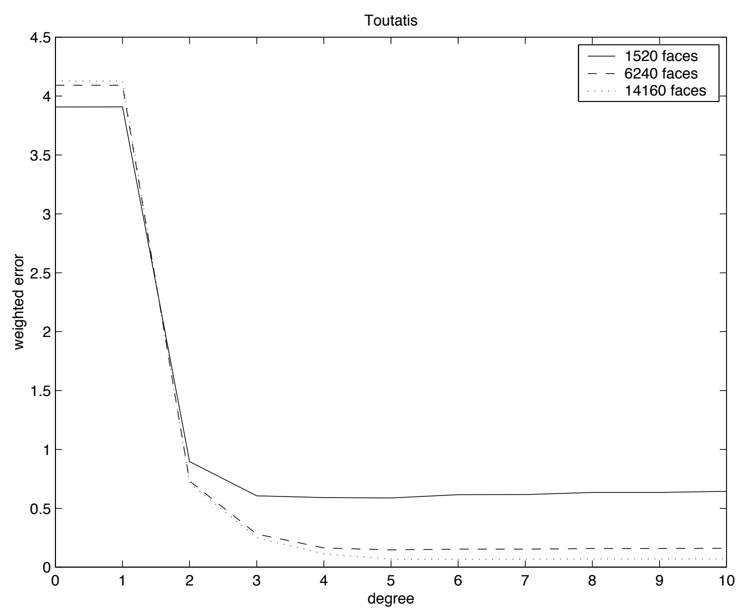


Figure 4.4: Weighted error for Toutatis

is not sufficient to accurately compute the coefficients α_{np}. To double check this statement, we numerically evaluate the coefficients A_{np}^{lm} and B_{np}^{lm} in the expression of the α_{np}. These evaluations are conducted for different levels of discretization of the "tessellated surface" E_{λ_1} and compared to the theoretical results summarized in Tables 2.1 and 2.2 of Chapter 2. We notice that these theoretical results are violated if the "tessellated surface" does not sufficiently match the ellipsoid E_{λ_1}. This now explains the erroneous behavior of the plots for the 1520 faces model.

The second observation we make is that the error tends to stabilize as we add higher degree terms in the ellipsoidal harmonics expansion instead of decreasing to zero. The ellipsoidal harmonics coefficients α_{np} have been computed from the spherical harmonics coefficients C_{lm} and S_{lm} up to a degree L:

$$\overline{\alpha_{np}} = \sum_{l=0}^{L} \sum_{m=0}^{l} (A_{np}^{lm} \cdot C_{lm} + B_{np}^{lm} \cdot S_{lm}). \tag{4.8}$$

Thus we expect the ellipsoidal harmonics expansion of the potential to reconstruct the spherical harmonics expansion, that is ϵ should go to 0 as we increase the degree n in the ellipsoidal harmonics expansion. The degree $n = 10$ may not be sufficient to match a 12 (Eros) or 16 (Toutatis) degree and order spherical harmonics expansion. This hypothethis appears to be true when reconstructing a 6 degree and order spherical harmonics expansion with a 12 degree and order ellipsoidal harmonics expansion (Figures 4.5 and 4.6).

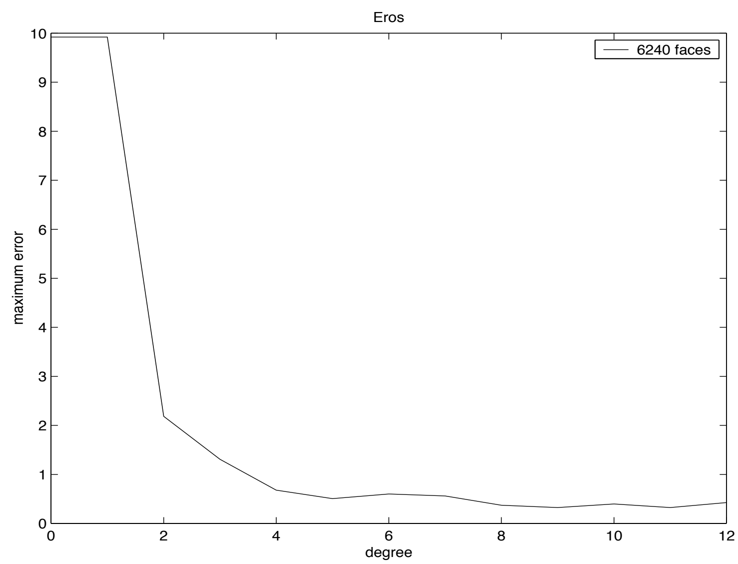


Figure 4.5: Maximum error for Eros with a 6 degree and order spherical harmonics expansion of the potential taken as the reference

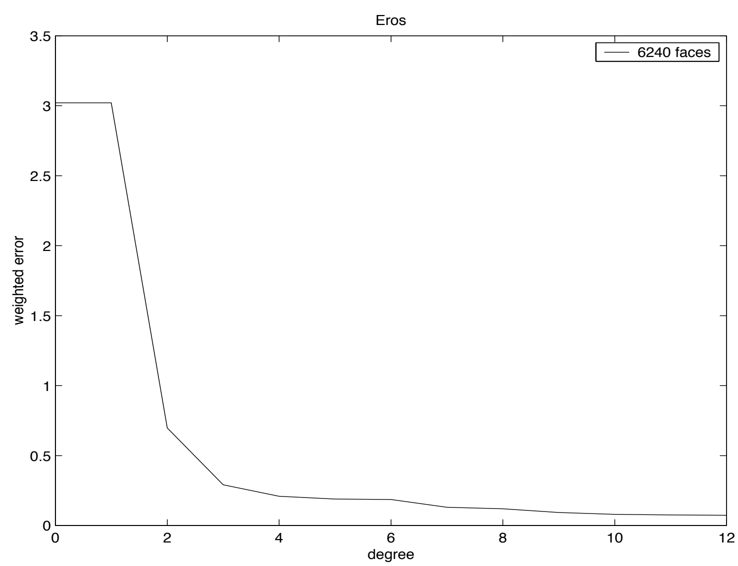


Figure 4.6: Weighted error for Eros with a 6 degree and order spherical harmonics expansion of the potential taken as the reference

Conclusion

Ellipsoidal harmonics are very suitable to asteroids and comets; however they are very complex. The goal of this thesis has been to use the knowledge on spherical harmonics to provide a better understanding of the ellipsoidal harmonics theory. Because of the obvious analogy between spherical and ellipsoidal harmonics, we have been interested in establishing an analytical expression that relates the spherical harmonics coefficients C_{lm} and S_{lm} to the ellipsoidal harmonics coefficients α_{np}. The coefficients α_{np} are the only terms in the expansion that incorporate "information" on the shape and density of the attracting body. In Chapter 2, we have taken advantage of an interesting orthogonalization property of the Lamé functions to express the α_{np} as a linear transformation of the C_{lm} and S_{lm}.

$$\overline{\alpha_{np}} = \sum_{l=0}^{\infty} \sum_{m=0}^{l} (A_{np}^{lm} \cdot C_{lm} + B_{np}^{lm} \cdot S_{lm}).$$

The coefficients A_{np}^{lm} and B_{np}^{lm} have been formulated as integrals over an ellipsoid. Although we have not been able to come up with explicit expressions for these coefficients, we have shown that the linear transformation $\overline{\alpha_{np}} = f(C_{lm}, S_{lm})$ could be further simplified based on symmetry arguments.

When numerically computing the A_{np}^{lm} and B_{np}^{lm} in Chapter 4, we have observed that the A_{np}^{lm} and B_{np}^{lm} vanish not only for the cases predicted in Chapter 2, but also for many other cases. For these cases further investigation needs to be conducted. The main problem that arises is that the integrand in the expressions of A_{np}^{lm} and B_{np}^{lm} is a function of both spherical and ellipsoidal coordinates with no obvious relation between these two sets. One approach that may be interesting to investigate is that the A_{np}^{lm} and B_{np}^{lm} are independent of the choice of the ellipsoid E_{λ_1}. In addition, for $\lambda_1 \to \infty$, the ellipsoid E_{λ_1} becomes a sphere.

Thus integrating over the surface of a sphere instead of an ellipsoid may lead to further progress regarding an analytical expression of the A_{np}^{lm} and B_{np}^{lm}.

Finally, for the same degree of expansion $l = n$, it is unclear which of the spherical or ellipsoidal harmonics expansion is more accurate. Both expansions involve the same number of coefficients. However, for some particular cases such as the homogeneous triaxial ellipsoid (see [8]), the gravity field is entirely described by a lot less ellipsoidal harmonics coefficients α_{np} than spherical harmonics coefficients C_{lm} and S_{lm}.

Appendix A: Lamé functions of the first kind of degree $n \leq 3$

$$E_0^1(x) = 1$$

$$E_1^1(x) = x$$

$$E_1^2(x) = \sqrt{|x^2 - h^2|}$$

$$E_1^3(x) = \sqrt{|x^2 - k^2|}$$

$$E_2^1(x) = x^2 - \frac{1}{3}\left[\,(h^2 + k^2) - \sqrt{(h^2 + k^2)^2 - 3h^2k^2}\,\right]$$

$$E_2^2(x) = x^2 - \frac{1}{3}\left[\,(h^2 + k^2) + \sqrt{(h^2 + k^2)^2 - 3h^2k^2}\,\right]$$

$$E_2^3(x) = x\,\sqrt{|x^2 - h^2|}$$

$$E_2^4(x) = x\,\sqrt{|x^2 - k^2|}$$

$$E_2^5(x) = \sqrt{|(x^2 - h^2)\,(x^2 - k^2)|}$$

$$E_3^1(x) = x^3 - \frac{x}{5}\left[\,2(h^2 + k^2) - \sqrt{4(h^2 + k^2)^2 - 15h^2k^2}\,\right]$$

$$E_3^2(x) = x^3 - \frac{x}{5}\left[\,2(h^2 + k^2) + \sqrt{4(h^2 + k^2)^2 - 15h^2k^2}\,\right]$$

$$E_3^3(x) = \sqrt{|x^2 - h^2|}\left(x^2 - \frac{1}{5}\left[\,h^2 + 2k^2 - \sqrt{(h^2 + 2k^2)^2 - 5h^2k^2}\,\right]\right)$$

$$E_3^4(x) = \sqrt{|x^2 - h^2|}\left(x^2 - \frac{1}{5}\left[\,h^2 + 2k^2 + \sqrt{(h^2 + 2k^2)^2 - 5h^2k^2}\,\right]\right)$$

$$E_3^5(x) = \sqrt{|x^2 - k^2|}\left(x^2 - \frac{1}{5}\left[\,2h^2 + k^2 - \sqrt{(2h^2 + k^2)^2 - 5h^2k^2}\,\right]\right)$$

$$E_3^6(x) = \sqrt{|x^2 - k^2|}\left(x^2 - \frac{1}{5}\left[\,2h^2 + k^2 + \sqrt{(2h^2 + k^2)^2 - 5h^2k^2}\,\right]\right)$$

$$E_3^7(x) = x\,\sqrt{|(x^2 - h^2)\,(x^2 - k^2)|}$$

Appendix B: Normalization constants of degree $n \le 3$

$$\gamma_0^1 = 4\,\pi$$

$$\gamma_1^1 = \frac{4\,\pi\,h^2\,k^2}{3}$$
$$\gamma_1^2 = \frac{4\,\pi\,h^2\,(k^2-h^2)}{3}$$
$$\gamma_1^3 = \frac{4\,\pi\,k^2\,(k^2-h^2)}{3}$$

$$\gamma_2^1 = \frac{16\,\pi}{405}\Big[\,2h^8+2k^8-4h^6k^2-4h^2k^6+6h^4k^4 + \sqrt{h^4+k^4-h^2k^2}(-2h^6-2k^6+3h^4k^2+3h^2k^4)\,\Big]$$
$$\gamma_2^2 = \frac{16\,\pi}{405}\Big[\,2h^8+2k^8-4h^6k^2-4h^2k^6+6h^4k^4 + \sqrt{h^4+k^4-h^2k^2}(2h^6+2k^6-3h^4k^2-3h^2k^4)\,\Big]$$
$$\gamma_2^3 = \frac{4\,\pi\,h^4\,k^2\,(k^2-h^2)}{15}$$
$$\gamma_2^4 = \frac{4\,\pi\,h^2\,k^4\,(k^2-h^2)}{15}$$
$$\gamma_2^5 = \frac{4\,\pi\,h^2\,k^2\,(k^2-h^2)^2}{15}$$

$$\gamma_3^1 = \frac{16\,\pi}{13125}\,h^2\,k^2\Big[\,16h^8+16k^8-36h^6k^2-36h^2k^6+46h^4k^4 + \sqrt{4h^4+4k^4-7h^2k^2}(-8h^6-8k^6+11h^4k^2+11h^2k^4)\,\Big]$$
$$\gamma_3^2 = \frac{16\,\pi}{13125}\,h^2\,k^2\Big[\,16h^8+16k^8-36h^6k^2-36h^2k^6+46h^4k^4 + \sqrt{4h^4+4k^4-7h^2k^2}(8h^6+8k^6-11h^4k^2-11h^2k^4)\,\Big]$$
$$\gamma_3^3 = \frac{16\,\pi}{13125}\,h^2\,(k^2-h^2)\Big[\,6h^8+16k^8-12h^6k^2-28h^2k^6+34h^4k^4 + \sqrt{h^4+4k^4-h^2k^2}(-6h^6-8k^6+9h^4k^2+13h^2k^4)\,\Big]$$
$$\gamma_3^4 = \frac{16\,\pi}{13125}\,h^2\,(k^2-h^2)\Big[\,6h^8+16k^8-12h^6k^2-28h^2k^6+34h^4k^4 + \sqrt{h^4+4k^4-h^2k^2}(6h^6+8k^6-9h^4k^2-13h^2k^4)\,\Big]$$
$$\gamma_3^5 = \frac{16\,\pi}{13125}\,k^2\,(k^2-h^2)\Big[\,16h^8+6k^8-28h^6k^2-12h^2k^6+34h^4k^4$$

$$+\sqrt{4h^4+k^4-h^2k^2}(-8h^6-6k^6+13h^4k^2+9h^2k^4)\,\big]$$

$$\gamma_3^6=\frac{16\,\pi}{13125}\,k^2\,(k^2-h^2)\,\big[\,16h^8+6k^8-28h^6k^2-12h^2k^6+34h^4k^4$$

$$+\sqrt{4h^4+k^4-h^2k^2}(8h^6+6k^6-13h^4k^2-9h^2k^4)\,\big]$$

$$\gamma_3^7=\frac{4\,\pi\,h^4\,k^4\,(k^2-h^2)^2}{105}$$

Appendix C: Leading products ψ, partials of $(\lambda_1, \lambda_2, \lambda_3)$ with respect to (x, y, z)

Leading products for Lamé functions of the first kind

$\boldsymbol{\psi_n^p(\lambda)}$		
$\overline{E_n^p}$	n even	n odd
type K	1	λ
type L	$\lambda\sqrt{\lvert\lambda^2-h^2\rvert}$	$\sqrt{\lvert\lambda^2-h^2\rvert}$
type M	$\lambda\sqrt{\lvert\lambda^2-k^2\rvert}$	$\sqrt{\lvert\lambda^2-k^2\rvert}$
type N	$\sqrt{\lvert(\lambda^2-h^2)(\lambda^2-k^2)\rvert}$	$\lambda\sqrt{\lvert(\lambda^2-h^2)(\lambda^2-k^2)\rvert}$

Expression of $\psi_n^p(\lambda_2)\psi_n^p(\lambda_3)$ in terms of the cartesian coordinates (x, y, z)

$\boldsymbol{\psi_n^p(\lambda_2)\psi_n^p(\lambda_3)}$		
$\overline{E_n^p}$	n even	n odd
type K	1	$\dfrac{hk}{\lambda_1}\,x$
type L	$\dfrac{h^2k}{\lambda_1}\sqrt{\dfrac{k^2-h^2}{\lambda_1^2-h^2}}\,xy$	$h\sqrt{\dfrac{k^2-h^2}{\lambda_1^2-h^2}}\,y$
type M	$\dfrac{hk^2}{\lambda_1}\sqrt{\dfrac{k^2-h^2}{\lambda_1^2-k^2}}\,xz$	$k\sqrt{\dfrac{k^2-h^2}{\lambda_1^2-k^2}}\,z$
type N	$\dfrac{hk(k^2-h^2)}{\sqrt{(\lambda_1^2-h^2)(\lambda_1^2-k^2)}}\,yz$	$\dfrac{h^2k^2(k^2-h^2)}{\lambda_1\sqrt{(\lambda_1^2-h^2)(\lambda_1^2-k^2)}}\,xyz$

Partial derivatives of the ellipsoidal coordinates $(\lambda_1, \lambda_2, \lambda_3)$ with respect to the cartesian coordinates (x, y, z)

$$\begin{pmatrix} \frac{\partial\lambda_1}{\partial x} & \frac{\partial\lambda_2}{\partial x} & \frac{\partial\lambda_3}{\partial x} \\ \frac{\partial\lambda_1}{\partial y} & \frac{\partial\lambda_2}{\partial y} & \frac{\partial\lambda_3}{\partial y} \\ \frac{\partial\lambda_1}{\partial z} & \frac{\partial\lambda_2}{\partial z} & \frac{\partial\lambda_3}{\partial z} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{(\lambda_1^2-k^2)(\lambda_1^2-h^2)}{\lambda_1(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}\,x & \frac{(\lambda_2^2-k^2)(\lambda_2^2-h^2)}{\lambda_2(\lambda_2^2-\lambda_1^2)(\lambda_2^2-\lambda_3^2)}\,x & \frac{(\lambda_3^2-k^2)(\lambda_3^2-h^2)}{\lambda_3(\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_2^2)}\,x \\ \frac{\lambda_1(\lambda_1^2-k^2)}{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}\,y & \frac{\lambda_2(\lambda_2^2-k^2)}{(\lambda_2^2-\lambda_1^2)(\lambda_2^2-\lambda_3^2)}\,y & \frac{\lambda_3(\lambda_3^2-k^2)}{(\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_2^2)}\,y \\ \frac{\lambda_1(\lambda_1^2-h^2)}{(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)}\,z & \frac{\lambda_2(\lambda_2^2-h^2)}{(\lambda_2^2-\lambda_1^2)(\lambda_2^2-\lambda_3^2)}\,z & \frac{\lambda_3(\lambda_3^2-h^2)}{(\lambda_3^2-\lambda_1^2)(\lambda_3^2-\lambda_2^2)}\,z \end{pmatrix}$$

Appendix D: Eigenvalue problem for the Lamé functions of the first kind

Coefficients a_i, b_i and c_i of the matrix A

Let

$$\sigma = \begin{cases} \frac{1}{2}\, n & \text{for } n \text{ even,} \\ \frac{1}{2}\,(n-1) & \text{for } n \text{ odd,} \end{cases}$$

and

$$\alpha = \big(\frac{h}{k}\big)^2, \quad \beta = 1 - \big(\frac{h}{k}\big)^2, \quad \gamma = \alpha - \beta.$$

The coefficients a_i, b_i, c_i are then given by:

– for K_n^p and n even,

$$\begin{cases} a_i = -(2\sigma - 2i + 2)(2\sigma + 2i - 1)\alpha \\ b_i = 2\sigma(2\sigma + 1)\alpha - 4i^2\gamma \\ c_i = -(2i + 1)(2i + 2)\beta \end{cases}$$

– for K_n^p and n odd,

$$\begin{cases} a_i = -(2\sigma - 2i + 2)(2\sigma + 2i + 1)\alpha \\ b_i = (2\sigma + 1)(2\sigma + 2)\alpha - 4i^2\alpha + (2i + 1)^2\beta \\ c_i = -(2i + 1)(2i + 2)\beta \end{cases}$$

– for L_n^p and n even,

$$\begin{cases} a_i = -(2\sigma - 2i)(2\sigma + 2i + 1)\alpha \\ b_i = 2\sigma(2\sigma + 1)\alpha - (2i + 1)^2\alpha + (2i + 2)^2\beta \\ c_i = -(2i + 2)(2i + 3)\beta \end{cases}$$

– for L_n^p and n odd,

$$\begin{cases} a_i = -(2\sigma - 2i + 2)(2\sigma + 2i + 1)\alpha \\ b_i = (2\sigma + 1)(2\sigma + 2)\alpha - (2i + 1)^2\gamma \\ c_i = -(2i + 2)(2i + 3)\beta \end{cases}$$

– for M_n^p and n even,

$$\begin{cases} a_i = -(2\sigma - 2i)(2\sigma + 2i + 1)\alpha \\ b_i = 2\sigma(2\sigma + 1)\alpha - (2i + 1)^2\gamma \\ c_i = -(2i + 1)(2i + 2)\beta \end{cases}$$

– for M_n^p and n odd,

$$\begin{cases} a_i = -(2\sigma - 2i + 2)(2\sigma + 2i + 1)\alpha \\ b_i = (2\sigma + 1)(2\sigma + 2)\alpha - (2i + 1)^2\alpha + 4i^2\beta \\ c_i = -(2i + 1)(2i + 2)\beta \end{cases}$$

– for N_n^p and n even,

$$\begin{cases} a_i = -(2\sigma - 2i)(2\sigma + 2i + 1)\alpha \\ b_i = 2\sigma(2\sigma + 1)\alpha - (2i + 2)^2\alpha + (2i + 1)^2\beta \\ c_i = -(2i + 2)(2i + 3)\beta \end{cases}$$

– for N_n^p and n odd,

$$\begin{cases} a_i = -(2\sigma - 2i)(2\sigma + 2i + 3)\alpha \\ b_i = (2\sigma + 1)(2\sigma + 2)\alpha - (2i + 2)^2\gamma \\ c_i = -(2i + 2)(2i + 3)\beta \end{cases}$$

Appendix E: Computation of the normalization constants γ_n^p

The $(E_n^p(\lambda))^2$ can be rewritten as a polynomial in Λ:

$$\Big(E_n^p(\lambda)\Big)^2 = \sum_{j=0}^{n} C_j\,\Lambda^j,$$

where the coefficients C_j's are related to the D_j's as follows:

– for K_n^p and n even, $\psi_n^p(\lambda) = 1$ and

$$C_j = D_j, \quad j = 0, \ldots, n$$

– for K_n^p and n odd, $\psi_n^p(\lambda) = \lambda$ and

$$\begin{cases} C_0 = h^2\, D_0 \\ C_j = h^2\,(D_j - D_{j-1}), \quad j = 1, \ldots, n-1 \\ C_n = -h^2\, D_{n-1} \end{cases}$$

– for L_n^p and n even, $\psi_n^p(\lambda) = \lambda\,\sqrt{|\lambda^2 - h^2|}$ and

$$\begin{cases} C_0 = 0 \\ C_1 = h^4\, D_0\, sign(\Lambda) \\ C_j = h^4\,(D_{j-1} - D_{j-2})\, sign(\Lambda), \quad j = 2, \ldots, n-1 \\ C_n = -h^4\, D_{n-2}\, sign(\Lambda) \end{cases}$$

– for L_n^p and n odd, $\psi_n^p(\lambda) = \sqrt{|\lambda^2 - h^2|}$ and

$$\begin{cases} C_0 = 0 \\ C_j = h^2\, D_{j-1}\, sign(\Lambda), \quad j = 1, \ldots, n \end{cases}$$

– for M_n^p and n even, $\psi_n^p(\lambda) = \lambda\,\sqrt{|\lambda^2 - k^2|}$ and

$$\begin{cases} C_0 \quad = h^2\,(k^2 - h^2)\, D_0 \\ C_1 \quad = h^2\,(k^2 - h^2)\, D_1 + h^2\,(2\,h^2 - k^2)\, D_0 \\ C_j \quad = h^2\,(k^2 - h^2)\, D_j + h^2\,(2h^2 - k^2)\, D_{j-1} - h^4\, D_{j-2}, \quad j = 2, \ldots, n-2 \\ C_{n-1} = h^2\,(2h^2 - k^2)\, D_{n-2} - h^4\, D_{n-3} \\ C_n \quad = -h^4\, D_{n-2} \end{cases}$$

– for M_n^p and n odd, $\psi_n^p(\lambda) = \sqrt{|\lambda^2 - k^2|}$ and

$$\begin{cases} C_0 = (k^2 - h^2)\, D_0 \\ C_j = (k^2 - h^2)\, D_j + h^2\, D_{j-1}, \quad j = 1, \ldots, n-1 \\ C_n = h^2\, D_{n-1} \end{cases}$$

– for N_n^p and n even, $\psi_n^p(\lambda) = \sqrt{|(\lambda^2 - h^2)(\lambda^2 - k^2)|}$ and

$$\begin{cases} C_0 = 0 \\ C_1 = h^2\,(k^2 - h^2)\, D_0\, sign(\Lambda) \\ C_j = \left[h^2\,(k^2 - h^2)\, D_{j-1} + h^4\, D_{j-2}\right] sign(\Lambda), \quad j = 2, \ldots, n-1 \\ C_n = h^4\, D_{n-2}\, sign(\Lambda) \end{cases}$$

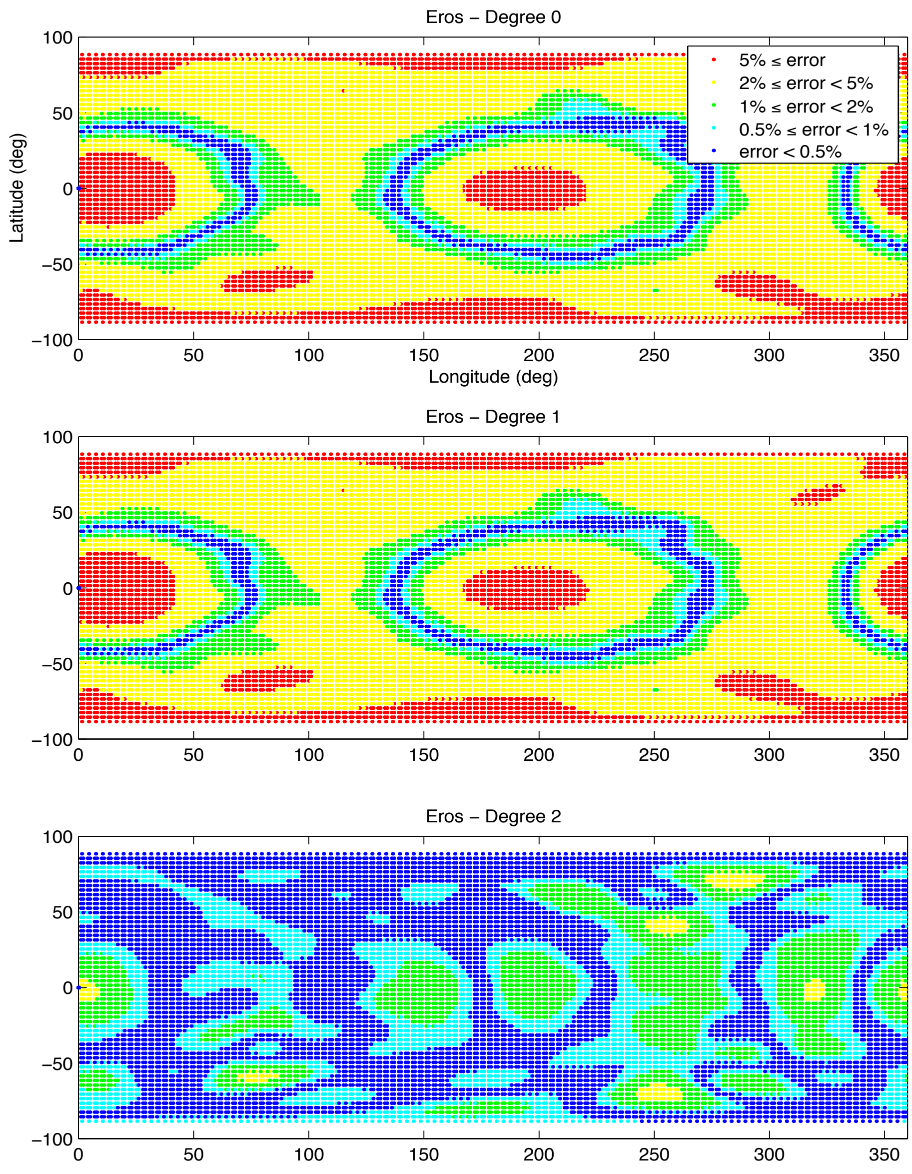
– for N_n^p and n odd, $\psi_n^p(\lambda) = \lambda\,\sqrt{|(\lambda^2 - h^2)(\lambda^2 - k^2)|}$ and

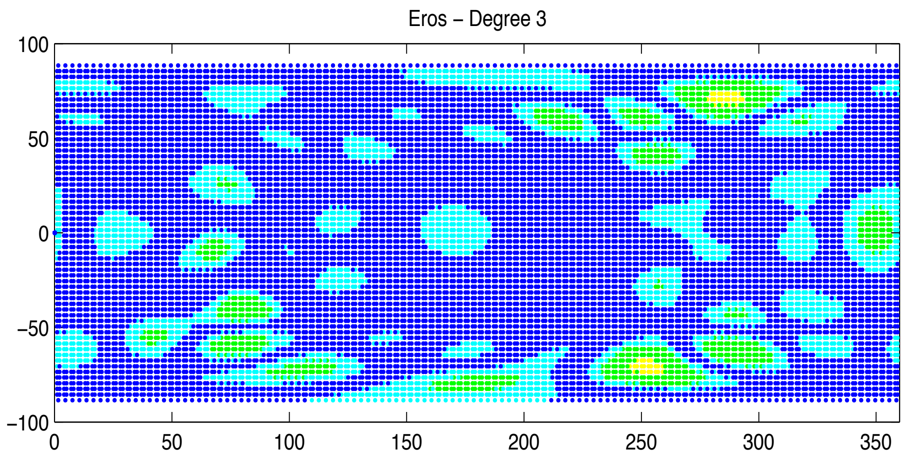
$$\begin{cases} C_0 \quad = 0 \\ C_1 \quad = h^4\,(k^2 - h^2)\, D_0\, sign(\Lambda) \\ C_2 \quad = \left[h^4\,(k^2 - h^2)\, D_1 + h^4\,(2h^2 - k^2)\, D_0\right] sign(\Lambda) \\ C_j \quad = \left[h^4\,(k^2 - h^2)\, D_{j-1} + h^4\,(2h^2 - k^2)\, D_{j-2} - h^6\, D_{j-3}\right] sign(\Lambda), \quad j = 3, \ldots, n-2 \\ C_{n-1} = \left[h^4\,(2h^2 - k^2)\, D_{n-3} - h^6\, D_{n-4}\right] sign(\Lambda) \\ C_n \quad = -h^6\, D_{n-3}\, sign(\Lambda) \end{cases}$$

In the preceding expressions,

$$sign(\Lambda) = \begin{cases} +1 & \text{if} \quad \lambda = \lambda_3, \\ -1 & \text{if} \quad \lambda = \lambda_2. \end{cases}$$

Appendix F: Accuracy of the ellipsoidal harmonics expansion for Eros





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0
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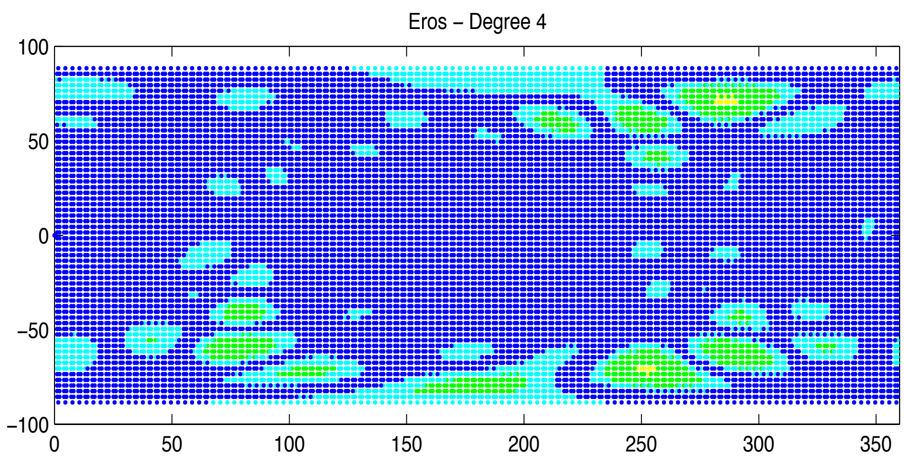
300
350



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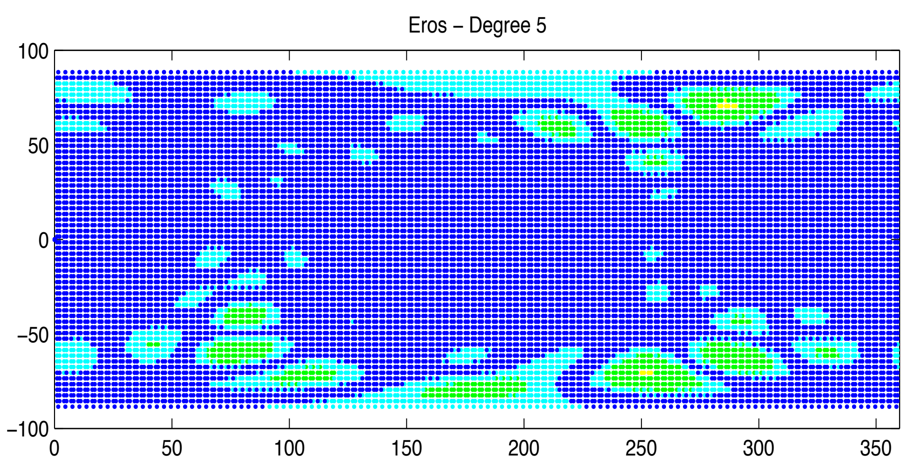

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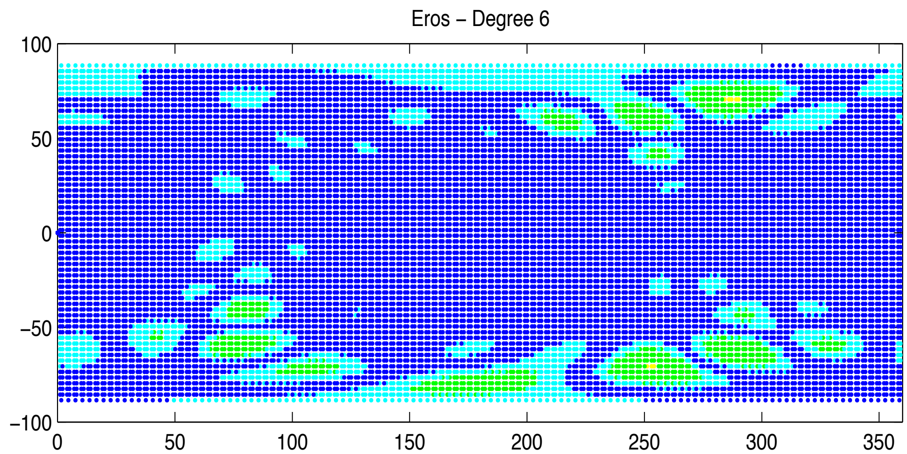
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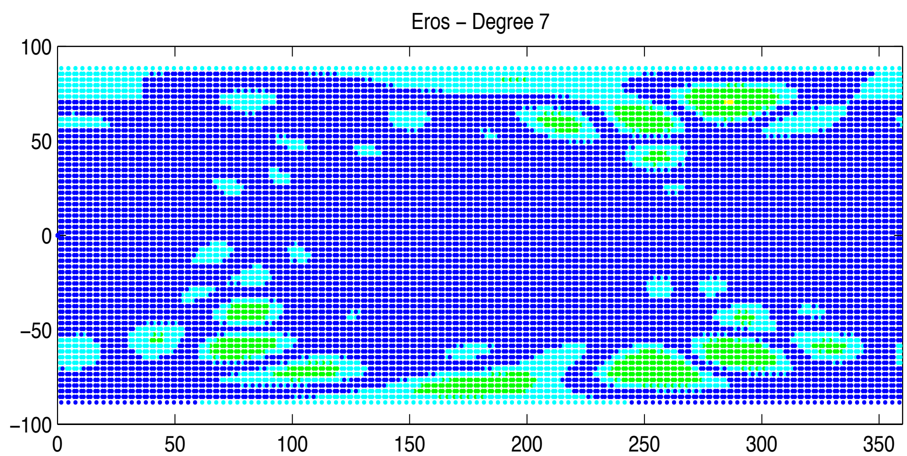
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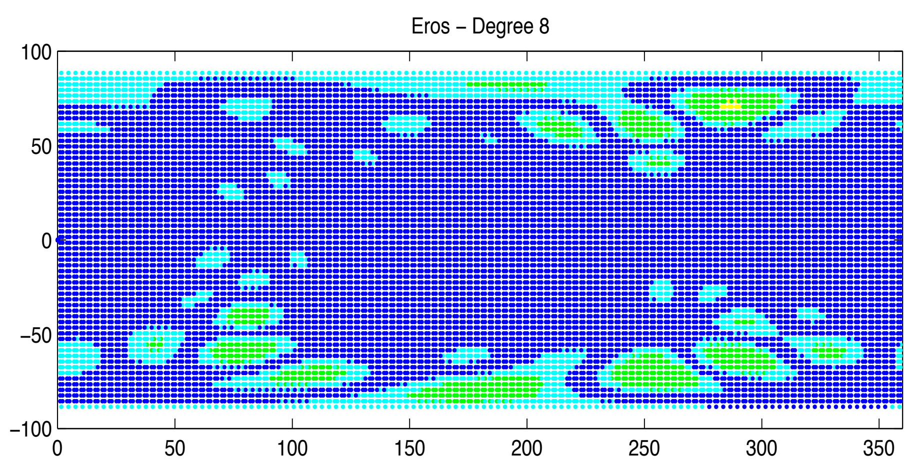
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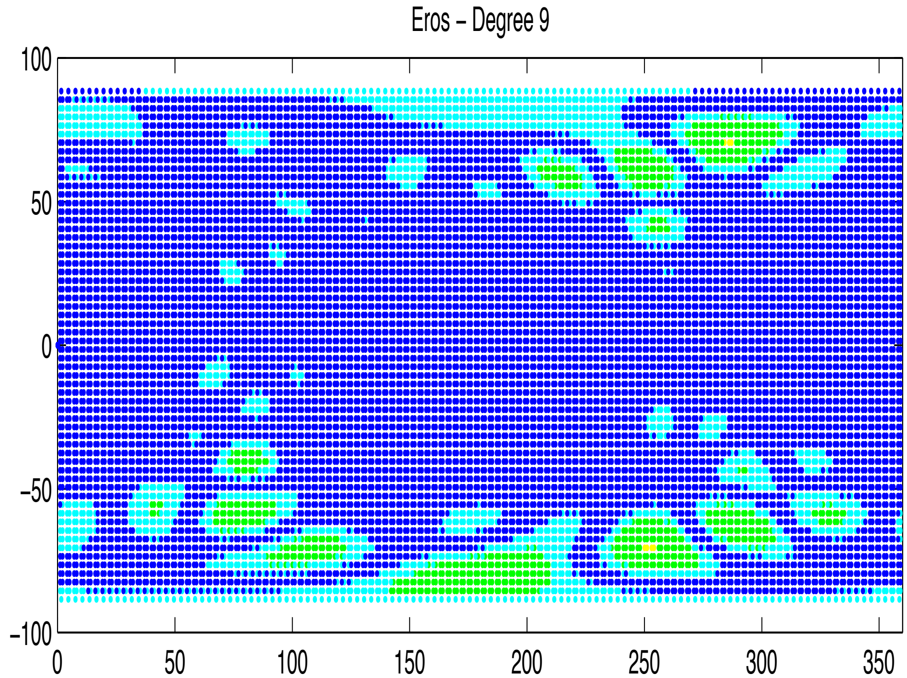
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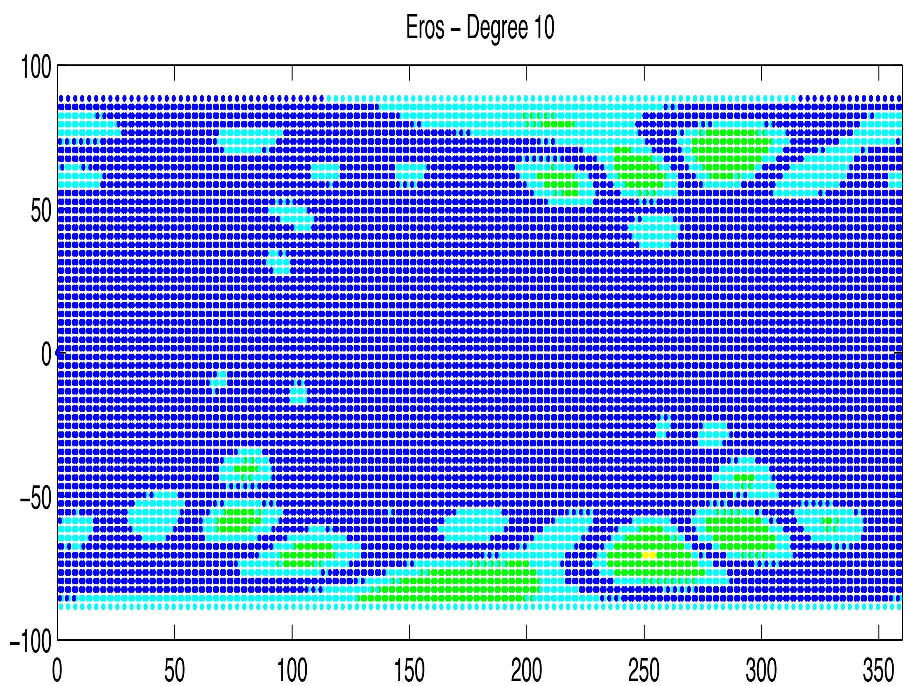
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