LOCALLY OPTIMAL TRANSFERS BETWEEN LIBRATION POINT ORBITS USING INVARIANT MANIFOLDS

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A method is developed for constructing locally optimal transfer trajectories between libration point orbits with different energies. The unstable manifold of the first orbit is connected to the stable manifold of the second orbit by the execution of two or more maneuvers. Two-body parameters define the selection of the unstable and stable manifold trajectories used for the transfer. The maneuver locations along the manifolds are determined by an application of primer vector theory. This method produces fuel costs up to 73% less than transfer trajectories that do not employ the use of manifolds.

INTRODUCTION

Libration point orbits have been a subject of interest for many years. Many missions are currently operating in libration point orbits, or are in the planning phases to send spacecraft, or constellations of spacecraft, to libration point orbits in the Sun-Earth and Earth-Moon systems. Halo and Lissajous orbits are planned as the nominal science orbits for several upcoming and proposed science missions, such as the James Webb Space Telescope and the X-Ray Evolving Universe Spectroscopy Mission. Additionally, halo orbits have been proposed for lunar navigation and communication relay constellations by many studies.1–4

With an increased interest in libration point orbits, an inexpensive method of constructing transfers between libration point orbits would be a powerful mission design tool. The ability to transfer spacecraft between libration point orbits would add a great deal of flexibility to a mission. Mission designers would be able to create transfers for spacecraft to travel to different orbits to achieve mission objectives, insert into a new orbit for a follow-on mission, or, in the case of a lunar communication relay, improve coverage characteristics for certain regions of the lunar surface.

The success of the Genesis mission illustrated the importance and benefits of incorporating techniques from dynamical systems theory into libration point orbit mission design. Previous research has been successful in developing techniques to transfer between libration point orbits of the same energy using invariant manifolds.5–8 Davis et al. developed a method to connect libration point orbits of different energies using invariant manifolds and a bridging trajectory.9 The transfers that were constructed were locating using a genetic algorithm that searched over a number of parameters to determine the cost. This research improves upon the method presented by Davis et al. by optimizing the transfers using an application of primer vector theory.

Primer vector theory was first developed by Lawden, who established the necessary conditions for optimal impulsive trajectories based on the primer vector and its derivative.10–13 The primer vector
is a term established by Lawden to refer to a vector comprised of the adjoint variables associated with the velocity of a particle on a trajectory. Lion and Handelsman extended Lawden’s work to develop conditions for improving a reference impulsive trajectory by the addition of an interior impulse or by the inclusion of initial and final coasts.\textsuperscript{14} Jezewski and Rozendaal devised a method of systematically locating the minimum cost for an $N$-impulsive trajectory.\textsuperscript{15} Jezewski provides a good summary of the contributions of Lawden and Lion and Handelsman to optimal trajectory theory and a good derivation of the equations associated with the primer vector.\textsuperscript{16}

Many researchers have utilized primer vector theory to locate optimal transfers. Two good examples are found in D’Amario,\textsuperscript{17} who used primer vector theory to compute optimal two- and three-impulse transfer trajectories between the Earth-Moon $L_2$ libration point and circular orbits about both the Earth and the Moon, and in the work of Hiday\textsuperscript{18} who extended Lawden’s work to develop the necessary conditions and transversality conditions for optimal trajectories with coastal arcs and interior impulses in the Elliptic-Restricted Three-Body Problem (ERTBP). The necessary conditions for an optimal trajectory in the Circular Restricted Three-Body Problem (CRTBP) are the same those in the ERTBP. A short summary of the necessary conditions for an optimal trajectory and select equations concerning the primer vector will be presented.

THE CIRCULAR RESTRICTED THREE-BODY PROBLEM

The motion of a spacecraft acting under the influence of two larger bodies can be modeled by the equations of motion of the Circular Restricted Three-Body Problem (CRTBP). The two primaries are assumed to rotate in circular orbits about the barycenter of the system. The reference frame rotates about the barycenter at the same rotation rate as the two primaries. The $x$-axis extends from the origin through the secondary, the $z$-axis extends in the direction of the angular momentum of the system, and the $y$-axis completes the right-hand coordinate frame. The equations describing the motion of the third body may be written as

\begin{align}
\ddot{x} &= 2\dot{y} + x - (1 - \mu) \frac{x + \mu}{R_1^3} - \mu \frac{x - 1 + \mu}{R_2^3}, \\
\ddot{y} &= -2\dot{x} + y - (1 - \mu) \frac{y}{R_1^3} - \mu \frac{y}{R_2^3}, \\
\ddot{z} &= -(1 - \mu) \frac{z}{R_1^3} - \mu \frac{z}{R_2^3},
\end{align}

where $R_1$ and $R_2$ are equal to the distance from the third body to the primary and secondary, respectively, and $\mu$ is the mass parameter used to nondimensionalize the system. The reader is directed to Szebehely\textsuperscript{19} for a derivation of the equations of motion.

The dynamics of the CRTBP allow the existence of an integral of motion to exist in the rotating frame. The equations of motion give by Equation 1 can be multiplied by $2\dot{x}$, $2\dot{y}$, and $2\dot{z}$, respectively, summed together, and integrated to obtain an integral of motion known as the Jacobi constant,

$$\mathcal{C} = 2\Omega - V^2,$$

where

$$\Omega = \frac{1}{2}(x^2 + y^2) + \frac{1 - \mu}{R_1} + \frac{\mu}{R_2}.$$

The Jacobi constant is analogous to energy in the two-body problem, in that the Jacobi constant cannot change unless the spacecraft is perturbed by something other than the two primaries. Note that it is only a function of the position and velocity magnitude expressed in the rotating frame.
Periodic orbits

The equations of motion of the CRTBP allow the existence of five equilibrium points, also known as the libration points. Families of periodic and quasi-periodic orbits exist about the libration points. Periodic orbits within the three-body problem have been studied by a multitude of researchers.\(^1\)\(^{20–28}\) There are three common classifications of unstable periodic orbits about libration points: Lissajous, halo, and Lyapunov orbits. A Lissajous orbit is a quasi-periodic orbital trajectory that winds around a torus, but never closes in on itself. Halo orbits are a special case of Lissajous orbits where the in-plane and out of plane frequencies are equal.\(^29\) Halo orbits are three-dimensional; a two-dimensional halo orbit is termed a Lyapunov orbit. Two common methods for numerically computing libration point orbits are a Richardson-Cary\(^30\) expansion and a Single-Shooting Algorithm.\(^27\)

Invariant manifolds

Unstable periodic orbits have an associated set of invariant manifolds. Invariant manifolds are trajectories that asymptotically depart or approach the orbit. The unstable manifold (\(W^U\)) includes the set of all possible trajectories that a particle on a nominal orbit could traverse if it was perturbed in the direction of the orbit’s unstable eigenvector. The stable manifold (\(W^S\)) includes the set of all possible trajectories that a particle could take to arrive onto the nominal orbit. Each orbit has two associated stable and unstable manifold sets: one corresponding to a positive perturbation, and one corresponding to a negative perturbation. Planar views of the unstable and stable manifolds of a halo orbit are shown in Figure 1. This paper will follow a common representation for the invariant manifolds: unstable manifold trajectories will be red and stable manifold trajectories will be green.

CONSTRUCTING A REFERENCE TRANSFER BETWEEN ORBITS

Davis et al.\(^9\) developed a method to construct transfer trajectories between libration point orbits using invariant manifolds. The method is briefly summarized here.

A bounding sphere, a sphere of radius \(R\) centered on the secondary, where \(R\) is less than the sphere of influence of the secondary, is used to locate transfer trajectories. Trajectories from the unstable manifold of the initial orbit and the stable manifold the final orbit are propagated to position intersections with the bounding sphere. Next, the trajectory piercings are integrated within the
sphere, such that successive points are spaced approximately equal in position. Next, the two-body parameters of the trajectories within the bounding sphere are analyzed. Because the radius of the bounding sphere is chosen to be less than the radius of the sphere of influence of the secondary, the portions of the trajectories that are contained within the bounding sphere are mainly influenced by the secondary. It was numerically demonstrated that two-body parameters could be used to analyze the unstable and stable manifold trajectories to help locate regions of low transfer costs. A two-body parameter, $\kappa$, is computed, where $\kappa$ is the sum of two quantities: the difference in the normalized angular momentum vectors between an unstable and a stable manifold trajectory, and the difference in the eccentricity vectors between an unstable manifold trajectory and a stable manifold trajectory. It was shown that as the two-body parameters of an unstable manifold trajectory more closely matched the two-body parameters of a stable manifold trajectory (i.e., unstable/stable manifold trajectory combinations with minimum $\kappa$ values), the total $\Delta V$ to complete the transfer decreased. Therefore, the specific unstable and stable manifold trajectories used for the transfer were selected based on the minimum $\kappa$ value. Two maneuvers were used to connect the unstable manifold of the initial orbit to the stable manifold of the final orbit. The first maneuver was performed on the unstable manifold and targeted a position on the stable manifold. After the execution of the first maneuver, a bridging trajectory was propagated to a position intersection with the stable manifold. The second maneuver was then executed to match the velocity on the bridging trajectory to the velocity on the stable manifold. A genetic algorithm was used to vary the parameters that defined the transfer (e.g., the maneuver locations along the unstable and stable manifolds).

In this research, the $\kappa$ parameter will still be used to determine the unstable/stable manifold pair for the transfer. However, the use of a genetic algorithm will be omitted, and the maneuver locations will be computed using primer vector theory.

**PRIMER VECTOR THEORY FORMULATION**

This section provides a brief description of the equations used in primer vector theory. Primer vector theory is employed to examine three methods that can lower the cost of a non-optimal two-impulse transfer trajectory: A coastal arc along the initial orbital trajectory (i.e., alter the location of the first maneuver), a coastal arc on the final orbital trajectory (i.e., alter the location of the final maneuver), and the inclusion of interior impulses. Hiday presents a comprehensive derivation of the formulation of the necessary conditions for an optimal trajectory in the ERTBP, as well as the conditions for when coastal arcs or interior impulses will lower the cost of a reference trajectory.\(^\text{18}\)

The equations of motion for the CRTBP can be expressed in vector format as

$$
\dot{\vec{R}} = \vec{V}
$$

$$
\dot{\vec{V}} = \frac{\beta c}{m} \hat{u}_T + \vec{g}
$$

$$
\dot{m} = -\beta,
$$

where

$$
\vec{g} = \begin{bmatrix}
2\dot{y} + \Omega_x \\
-2\dot{x} + \Omega_y \\
\Omega_z
\end{bmatrix},
$$

$m$ is the vehicle mass, and $\hat{u}_T$ is a unit vector in the thrusting direction with the constraint $\hat{u}_T^T \hat{u}_T = 1$. The magnitude of the thrust is given by $\beta c$, where $\beta$ is the mass flow rate and $c$ is the effective exhaust velocity.
For impulsive transfers where the goal is fuel minimization, the most logical choice for the performance index is the fuel expenditure. Thus, the optimality criteria can be based on the sum of the maneuvers necessary to complete the transfer,

\[ J = \sum_{i=1}^{N} |\Delta V_i|. \]  

(5)

Adjoint variables are introduced in the form of Lagrange multipliers so the state equations can be treated as differential constraints. The adjoint variables are defined as

\[ \lambda \triangleq \begin{bmatrix} \lambda_R \\ \lambda_V \\ \lambda_m \end{bmatrix}, \]

(6)

where \( \lambda_R \) and \( \lambda_V \) are vectors denoting the adjoints to the spacecraft’s position and velocity, respectively, and \( \lambda_m \) is a scalar quantity representing the adjoint to the spacecraft’s mass. Next, a Hamiltonian, \( H \), for the system can be formulated by multiplying \( \lambda^T \) by the right hand sides of the equations given in Equation 4,

\[ H = \lambda_R^T \dot{R} + \lambda_V^T \left( \frac{\beta c}{m} \dot{u} + \bar{g} \right) - \lambda_m \beta. \]  

(7)

The differential equations for the adjoints are determined by taking the negative partial derivatives of \( H \) with respect to each state:

\[
\begin{align*}
\dot{\lambda}_R &= -\frac{\partial H}{\partial R} = -\lambda_V^T G_R \\
\dot{\lambda}_V &= -\frac{\partial H}{\partial V} = -\lambda_R^T - \lambda_V^T G_V \\
\lambda_m &= -\frac{\partial H}{\partial m} = \frac{\beta c}{m^2} \lambda_V \dot{u}.
\end{align*}
\]

(8)

The matrices \( G_R \) and \( G_V \) are the partials of \( \bar{g} \) with respect to \( R \) and \( V \), respectively,

\[
G_R = \frac{\partial \bar{g}}{\partial R} = \begin{bmatrix} \Omega_{xx} & \Omega_{xy} & \Omega_{xz} \\ \Omega_{gx} & \Omega_{gy} & \Omega_{gz} \\ \Omega_{zx} & \Omega_{zy} & \Omega_{zz} \end{bmatrix},
\]

\[
G_V = \frac{\partial \bar{g}}{\partial V} = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

(9)

where \( \Omega \) is given in Equation 3. Note that in the CRTBP, the \( G_V \) matrix is different than in the ERTBP due to the omission of the eccentricity effects.

Henceforth, the adjoint vector to the velocity, \( \lambda_V \), will be called the primer vector, \( p \), and define \( \bar{p} = \lambda_V \).

The necessary conditions for an optimal trajectory can be expressed in terms of the primer vector:

1. The primer vector is continuous and has a continuous first derivative.
2. The primer vector satisfies the equation:

\[ \ddot{\mathbf{p}} = G_R \mathbf{p} + G_V \dot{\mathbf{p}}. \]  \hspace{1cm} (10)

3. The primer vector is a unit vector aligned in the optimal thrust direction at an impulse.

4. The magnitude of the primer vector equals unity at an impulse and has a value less than unity at all other instances along the transfer.

5. At all interior impulses \( \dot{\mathbf{p}} = 0 \), and the derivative of the primer vector and the primer must be orthogonal due to the continuity of the Hamiltonian at a corner.

The primer vector satisfies Equation 10 with the boundary conditions

\[ \mathbf{p}_0 = \frac{\Delta \mathbf{V}_0}{\mathbf{V}_0} \quad \text{and} \quad \mathbf{p}_f = \frac{\Delta \mathbf{V}_f}{\mathbf{V}_f}. \]  \hspace{1cm} (11)

In other words, the primer vector is a unit vector aligned in the velocity direction at the terminal endpoints on the trajectory. The quantities \( \Delta \mathbf{V}_0 \) and \( \Delta \mathbf{V}_f \) denote the impulsive maneuvers performed at the trajectory endpoints. One can compute the value for the primer vector at all points along the trajectory by propagating the primer vector using the state transition matrix

\[
\begin{bmatrix}
\mathbf{p}(t) \\
\dot{\mathbf{p}}(t)
\end{bmatrix} =
\begin{bmatrix}
\Phi_{11}(t, t_0) & \Phi_{12}(t, t_0) \\
\Phi_{21}(t, t_0) & \Phi_{22}(t, t_0)
\end{bmatrix}
\begin{bmatrix}
\mathbf{p}(t_0) \\
\dot{\mathbf{p}}(t_0)
\end{bmatrix}.
\]  \hspace{1cm} (12)

The value for \( \dot{\mathbf{p}}_0 \) can be found by noting that the primer vector is unity at the endpoints and using the relationship

\[ \dot{\mathbf{p}}(t_0) = \Phi_{12}^{-1}(t_f, t_0) \left[ \mathbf{p}(t_f) - \Phi_{11}(t_f, t_0) \mathbf{p}(t_0) \right]. \]  \hspace{1cm} (13)

A relationship can be established between contemporaneous variations in position and velocity (i.e., \( \delta \mathbf{R} \) and \( \delta \mathbf{V} \)) and the primer vector,

\[ \mathbf{p}^T \delta \mathbf{V} - \left( \mathbf{p}^T + \mathbf{p}^T G_V \right) \delta \mathbf{R} = \text{Constant}. \]  \hspace{1cm} (14)

This equation is known as the adjoint equation and it is valid over any interval between two impulses.

For a given reference two-impulse trajectory, the primer vector magnitude history is plotted over the duration of the transfer. If the conditions for an optimal trajectory are not met, then primer vector theory is used to optimize the transfer. First, the maneuver locations along the initial and final orbits are altered by the inclusion of coastal arcs to render the best two-impulse transfer trajectory. The resultant primer vector history is examined and if the trajectory is still not optimal, interior impulses are included to locate an optimal trajectory.

**Coastal Arcs**

Coastal arcs are employed to lower the cost of a reference two-impulse transfer trajectory. A coastal arc will move a maneuver in time, and subsequently, in position along its corresponding orbit. In this research, the initial maneuver is moved along the unstable manifold of the first orbit, and the final maneuver is moved along the stable manifold of the second orbit. Hiday-
Howell\textsuperscript{31} found the difference in cost between a reference trajectory and a trajectory with initial and final coastal arcs can be expressed as

\[ \delta J = -\dot{p}_0 |\Delta \nabla_0| dt_0 - \dot{p}_f |\Delta \nabla_f| dt_f. \quad (15) \]

A perturbed trajectory will have a lower cost than the reference trajectory if the quantity \( \delta J \) is negative, hence

\[ -\dot{p}_0 |\Delta \nabla_0| dt_0 - \dot{p}_f |\Delta \nabla_f| dt_f < 0 \quad (16) \]

For a given trajectory, the values of \( \dot{p}_0 \) and \( \dot{p}_f \) are known at the terminals. Consequently, these values dictate the signs of the coastal periods at the initial and final impulses. The following four cases determine the signs for the initial and final coastal arcs:

1. \( \dot{p}_0 > 0 \) and \( \dot{p}_f > 0 \) \( \Rightarrow \) \( dt_0 > 0 \) and \( dt_f > 0 \) \( \Rightarrow \) Initial Coast/Late Arrival.
2. \( \dot{p}_0 > 0 \) and \( \dot{p}_f < 0 \) \( \Rightarrow \) \( dt_0 > 0 \) and \( dt_f < 0 \) \( \Rightarrow \) Initial Coast/Final Coast.
3. \( \dot{p}_0 < 0 \) and \( \dot{p}_f > 0 \) \( \Rightarrow \) \( dt_0 < 0 \) and \( dt_f > 0 \) \( \Rightarrow \) Early Departure/Late Arrival.
4. \( \dot{p}_0 < 0 \) and \( \dot{p}_f < 0 \) \( \Rightarrow \) \( dt_0 < 0 \) and \( dt_f < 0 \) \( \Rightarrow \) Early Departure/Final Coast.

The signs of the primer magnitude slopes at the terminals (\( \dot{p}_0 \) and \( \dot{p}_f \)) are used to adjust the times of the coastal arcs accordingly (\( dt_0 \) and \( dt_f \)). The magnitude of the adjustment to the coastal arc time is proportional to the value of the terminal slope. A larger absolute value of the slope corresponds to a larger adjustment to the time of the initial or final coastal arc. After the coastal arcs have been adjusted, the slopes of the primer magnitude at the initial and final impulses are again computed and the process of adjusting the coastal arcs is repeated until the slopes at the terminals are close to zero, with a desired tolerance. This process can be implemented in an iterative fashion to quickly determine the coastal arcs that minimize the overall cost of the two-impulse transfer. The plot of the primer magnitude for a minimum two-impulse transfer should show primer magnitude values of one and slopes of zero at the terminal endpoints. For an optimal trajectory, the primer magnitude should never exceed unity. If the cost of a two-impulse transfer has been minimized such that the slopes are zero at the terminals, but the primer exceeds unity, the transfer is not optimal and interior impulses will decrease the cost of the transfer.

**Addition of Interior Impulses**

Given a reference non-optimal two-impulse transfer with zero slopes at the terminals, the greatest improvement in cost, to the first order, can be achieved by executing an interior impulse at the point where the primer vector magnitude is at a maximum. Additionally, the maneuver should be applied in the direction of the primer vector. Hiday-Johnston and Howell provide a detailed description of the equations required to compute the location and magnitude of the interior impulse. They also describe how to move the interior impulse, if the resulting three-impulse trajectory does not meet some of the criteria for an optimal trajectory.\textsuperscript{32}
TRANSFERS IN THE EARTH-MOON SYSTEM

Two halo orbits about the Earth-Moon L₂ point were selected to connect with a transfer trajectory. The initial orbit was chosen as a Southern halo with Jacobi constant of \( C = 3.0327 \) and a period of 7.5 days. The final halo orbit was selected to be a Southern halo orbit with a Jacobi constant of 3.06 and a period of 13.6 days. As a spacecraft would travel from a smaller orbit to a larger orbit, the transfer is called a superior transfer. Conversely, an inferior transfer is one that travels from a larger orbit to a smaller orbit. The initial orbit was designed by Hamra et al. for use in a lunar nav/comm relay and was optimized for South Pole coverage. The final orbit was designed by Hill et al. for a lunar nav/comm and gravity mission and has excellent far-side coverage.

First, a transfer trajectory will be constructed between the orbits that does not use invariant manifolds. It will later be compared to the transfer trajectory that does use invariant manifolds.

The impulse locations on the initial and final orbits were arbitrarily selected as the maximum \( z \)-amplitudes of each halo orbit. The first maneuver targeted the position vector on the final orbit corresponding to the location of the second maneuver. The magnitude and components of the first impulse were computed using a Level 1 differential corrector. The second maneuver corrected the velocity to match the velocity on the second orbit at the position intersection. The bridging trajectory was propagated under the equations of motion of the CRTBP. The primer vector magnitude history and the corresponding trajectory for the maneuver locations at the maximum \( z \)-amplitudes are presented in Figures 2(a) and 2(b), respectively.

![Figure 2](image)

Figure 2  The primer vector magnitude history and the corresponding trajectory for the reference two-impulse transfer.

The primer history shows a transfer that is clearly non-optimal. The slopes are non-zero at the terminals and the primer exceeds unity. Furthermore, the trajectory itself, Figure 2(b), shows a transfer where the first maneuver of 1554.5 m/s acts almost perpendicularly to the velocity direction to target the position on the final orbit. The final impulse of 565.5 m/s also acts in a nearly normal direction to the velocity.
The primer magnitude illustrated in Figure 2(a) has an initial slope that is positive, which points to an initial coast with \( dt_0 > 0 \), and it possesses a positive final slope, which implies that a late arrival, or coastal arc with \( dt_f > 0 \) is needed. The coastal periods that minimized the cost of the two-impulse transfer were computed and found to be 1.07 days and 4.51 days, respectively. The best two-maneuver primer history is presented in Figure 3. The distance to the center of mass of the secondary has been superimposed over the primer history in Figure 3(b) to show that the primer magnitude peaks very near the closest approach to the secondary. The resulting trajectory given in Figure 4 is a vast improvement over the reference trajectory. The initial and final coastal arcs substantially lowered the total cost from over 2200 m/s down to 117.01 m/s. The new maneuvers were found to be 48.00 m/s and 69.01 m/s. Note also that the time of flight has greatly increased. The time of flight for the transfer in Figure 2(b) was approximately 3 days while the best two-impulse transfer has a time of flight of 12.68 days.

Figure 3 The best primer vector magnitude history for the Earth-Moon system halo-to-halo orbit transfer with two maneuvers and no invariant manifolds. The transfer is not optimal. The total \( \Delta V = 117.01 \) m/s.

The best two-impulse transfer is non-optimal; the primer vector exceeds unity near the flyby of the secondary. The inclusion of an interior impulse will lower the cost of the trajectory. An interior maneuver was added at the location where the primer magnitude peaked. The resulting primer vector history is presented in Figure 5. The primer history satisfies all of the criteria for an optimal trajectory. The primer and its derivative are continuous at the interior impulse and the primer magnitude is less than or equal to unity at all instances.

Two views of the final optimal trajectory are given in Figure 6. The interior impulse is executed near the closest approach to the secondary. Examining the \( y-z \) transfer view in Figure 6(b) shows that the transfer starts near the maximum \( y \)-excursion on the initial orbit and ends a little after the maximum \( y \)-excursion on the final orbit. The transfer appears to perform one pseudo-revolution of a trajectory that connects the two orbits. The addition of the interior impulse decreased the total cost of the transfer from 117.01 m/s to 113.20 m/s.

Now the transfer constructed without manifolds will be compared to a transfer trajectory that has been constructed by using manifolds. The trajectories from the unstable manifold of the first orbit and the stable manifold of the second orbit were propagated to their intersections with the bounding sphere and the \( \kappa \) parameters of each unstable/stable manifold pair were computed within the
Figure 4  The trajectory for the best two impulse transfer in the Earth-Moon system with no invariant manifolds. The maneuvers are denoted by the yellow dots and the arrow indicates the direction of motion. The total $\Delta V = 117.01$ m/s.

Figure 5  The optimal primer vector magnitude history for the Earth-Moon system halo-to-halo orbit transfer with no invariant manifolds. The interior impulse is denoted by the yellow dot. The total $\Delta V = 113.20$ m/s.

bounding sphere. The unstable/stable manifold pair with the smallest $\kappa$ value was located and those manifold trajectories were selected for the transfer. A reference two-impulse transfer trajectory was created. The initial and final maneuver locations were selected to be approximately 10 hours before and 10 hours after the closest approach in position between the manifolds. The primer vector magnitude history was computed along the bridging trajectory that connected the unstable manifold of the first orbit to the stable manifold of the second orbit. The resultant plot is shown in Figure 7. The total $\Delta V$ for this transfer was 62.44 m/s. Although this transfer has not been optimized, it is a drastic decrease from the transfer that was constructed without manifolds (total $\Delta V = 113.20$ m/s).

The transfer is clearly non-optimal. The primer exceeds unity and the primer magnitude does not display zero slopes at both terminals. Initial and final coastal arcs will be employed to locate the best two-impulse transfer trajectory. The initial slope is positive, which indicates a coastal arc with
Figure 6 The trajectory for the optimal halo-to-halo orbit transfer in the Earth-Moon system with no invariant manifolds. The maneuvers are denoted by the yellow dots and the arrows indicate the direction of motion. The total $\Delta V = 113.20$ m/s.

Figure 7 The primer vector magnitude history of the reference two-impulse transfer trajectory for the halo-to-halo orbit transfer in the Earth-Moon system. The transfer is not optimal. The total $\Delta V$ is 62.44 m/s.

d$t_0 > 0$, and the final slope is also positive, which indicates a coastal arc with $dt_f > 0$ (i.e., a Late Arrival).

The coastal arcs were altered by approximately 8.1 hours and 2.13 days, respectively. A new primer vector magnitude history plot was created and is presented in Figure 8.

Figure 8 shows a non-optimal primer history. Although the slopes at the endpoints are now zero, the primer exceeds unity, violating the necessary conditions for an optimal trajectory. Figure 8(b) overlays the distance to the center of mass of the secondary on top of the primer vector history. This shows that the peak in the primer vector corresponds to a flyby of the Moon. The inclusion of the coastal arcs drastically reduced the total $\Delta V$ of the transfer. The best two-impulse transfer cost was 35.72 m/s, an improvement of over 26 m/s.

An interior impulse was added at the location where the primer magnitude achieved its maximum value. The resulting primer history is plotted in Figure 9. Figure 9(b) zooms in near the interior
Figure 8 The primer vector magnitude history of the best two-impulse transfer trajectory for the halo-to-halo orbit transfer in the Earth-Moon system. The transfer is not optimal. The total $\Delta V$ is 35.72 m/s.

Figure 9 The optimal primer vector magnitude history for the Earth-Moon system halo-to-halo orbit transfer. The total $\Delta V = 35.53$ m/s.

Figure 11 presents planar views of the reference two-impulse transfer and the optimal transfer to illustrate how the maneuver locations have been significantly changed to produce a large decrease in total cost.

The total cost of the optimal trajectory using invariant manifolds is over three times less expensive than the cost of the transfer trajectory that did not use manifolds. With the decreased cost came an increase in the time of flight. The time of flight for the optimal transfer with manifolds is 104
Figure 10  The optimal transfer trajectory for the Earth-Moon system halo-to-halo orbit transfer. The yellow dots denote the maneuver locations and the gray dot denotes the Moon. The total $\Delta V$ is 35.53 m/s.

Figure 11  Planar views of the reference two-impulse transfer trajectory and the optimal transfer trajectory. The yellow dots denote the maneuver locations.

days. However, as shown in Figure 10, a large portion of the time of flight is spent asymptotically departing the initial orbit. Recall that the initial and final halo orbits were originally designed for a lunar navigation and communication relay and were selected for their lunar coverage characteristics. As the majority of the trajectory stays on the far-side of the Moon in the vicinity of the initial orbit, this path provides excellent coverage of the South Pole and far-side of the Moon during the duration of transfer. Hence, the longer time of flight is not a hindrance to mission design.

The optimal transfer trajectory was plotted in an Earth-centered inertial frame and is shown in Figure 12. The Moon’s orbit is shown in blue. As seen in Figure 12, the transfer trajectory is almost always on the far-side of the Moon. There are a few instances where the trajectory crosses the Moon’s orbit. However, these times are short lived. This is another verification that the trajectory remains on the far-side of the Moon during the transfer, which would translate into excellent far-side coverage for a relay constellation mission.
Figure 12  The optimal transfer trajectory for the Earth-Moon system halo-to-halo orbit transfer plotted in an Earth-centered inertial reference frame. The yellow dots denote the maneuver locations and the gray dot denotes the Earth. The Moon’s orbit is denoted by the blue line.

Despite the fact that this transfer satisfies the conditions for an optimal trajectory, the transfer may not represent the global minimum in terms of the total $\Delta V$ cost. Consider the fact that the transfers illustrated in Figures 6 and 10 both satisfied the necessary conditions for an optimal trajectory. However, the cost for the transfer that does employ the use of invariant manifolds is only 35.5 m/s, over three times less than the transfer that did not use invariant manifolds. A transfer trajectory with a lower $\Delta V$ may be achieved by propagating the manifolds for a longer duration. Hence, both trajectories should be called locally optimal.

Now consider the reverse transfer, i.e., the path from the larger halo orbit to the smaller halo orbit. For transfers involving invariant manifolds, if a transfer has already been constructed to connect Orbit A to Orbit B, finding the transfer from Orbit B to Orbit A can be quite simple. There is an inherent symmetry of the invariant manifolds in the CRTBP about the $x$-axis. This can be seen in Figures 1(a) and 1(b). If a transfer from Orbit A to Orbit B is known, the transfer from Orbit B to Orbit A can be found by reflecting the original transfer about the $x$-axis. The first step in computing a trajectory that has been reflected about the $x$-axis is to manipulate the locations along the orbits that define the manifolds. Let the parameter $\tau$ describe the position of a particle on a periodic orbit where the value of $\tau$ ranges from 0–$2\pi$ radians, increasing in the direction of orbital motion. $\tau$ is defined to be zero at the orbit’s intersection with the $x$-$z$ plane in the $+\hat{y}$ direction, and $\pi$ radians at the intersection with the $x$-$z$ plane in the $-\hat{y}$ direction. The parameter $\tau$ can be used to define the starting locations of the manifolds along the orbits. To reflect the transfer about the $x$-axis, the $\tau$ values are altered as follows

$$\tau_{U_{B \to A}} = 2\pi - \tau_{S_{A \to B}} \quad \quad (17)$$

$$\tau_{S_{B \to A}} = 2\pi - \tau_{U_{A \to B}}. \quad \quad (18)$$

Equations 17 and 18 simply state that to find the new $\tau$ values for the transfer from Orbit B to Orbit A, subtract the initial $\tau$ values from $2\pi$, noting that the new $\tau_U$ is based on the old $\tau_S$ and vice versa.
In a similar fashion, the new times to propagate the manifolds can be found as

\begin{align*}
    t_{UB}^{A} &= t_{SA}^{B} \
    t_{SB}^{A} &= t_{UA}^{B} 
\end{align*}

(19)

(20)

It should be noted that Equations 19 and 20 give a first guess at the times to propagate the manifolds. It is likely that the new times will have to be optimized using primer vector theory.

Now, a trajectory will be constructed to transfer the spacecraft back from the larger halo orbit to the smaller halo orbit. The resulting optimal primer history and transfer trajectory are shown in Figures 13(a) and 13(b). Three maneuvers were required for an optimal transfer. The interior impulse was executed near the closest approach to the secondary. The primer history shown in Figure 13(a) is a reflection of the primer history given in Figure 9. Although it is difficult to see when comparing Figures 10 and 13(b), the trajectories are symmetric about the x-axis. The superior transfer spends most of its time of flight on the unstable manifold of the initial orbit (the smaller halo orbit), while the inferior transfer spends most of its time of flight on the stable manifold of the final orbit (again, the smaller halo orbit). The superior transfer had a total \( \Delta V \) of 35.53 m/s while the inferior transfer had a total \( \Delta V \) of 35.57 m/s.

\begin{figure}
\centering
\subfloat[]{
\includegraphics[width=0.45\textwidth]{figure13a.png}
\caption{The primer vector magnitude history and the transfer trajectory for the optimal inferior Earth-Moon system halo-to-halo orbit transfer. The yellow dots denote the maneuver locations and the gray dot denotes the Moon. The total \( \Delta V \) = 35.57 m/s.}
\label{fig:transfer13a}
}
\subfloat[]{
\includegraphics[width=0.45\textwidth]{figure13b.png}
\caption{The primer vector magnitude history and the transfer trajectory for the optimal inferior Earth-Moon system halo-to-halo orbit transfer. The yellow dots denote the maneuver locations and the gray dot denotes the Moon. The total \( \Delta V \) = 35.57 m/s.}
\label{fig:transfer13b}
}
\end{figure}

**TRANSFERS IN THE JUPITER EUROPA SYSTEM**

The Jupiter-Europa system offers an interesting array of libration point orbits and unstable periodic orbits (UPOs). Recently, Russell located families of unstable periodic orbits about Europa.\(^{33}\) These unstable periodic orbits will have associated invariant manifolds, and can thus be connected to other unstable periodic orbits using the techniques described in this paper. For a transfer in the Jupiter-Europa system, the initial orbit was selected as a halo orbit about L\(_{2}\) with Jacobi constant \( C = 3.001276 \). The final orbit was chosen to be an unstable periodic orbit about Europa with Jacobi constant \( C = 3.000769 \). The initial and final orbits are shown in Figure 14.
Figure 14. Views of the Unstable Periodic Orbit (UPO) about Europa and the L₂ halo orbit.

The unstable/stable manifold pair with the smallest $\kappa$ parameter was selected for the transfer. The initial and final maneuver locations for the reference two-impulse transfer trajectory were chosen to be approximately 12 hours before and after the closest position approach of the manifolds. The primer vector magnitude history was computed along the bridging trajectory that connected the unstable manifold of the first orbit to the stable manifold of the second orbit. The resultant plot is shown in Figure 15(a). The cost of the reference two-impulse trajectory is approximately 99 m/s.

The reference two-impulse primer history is clearly not optimal, as the slopes at the terminals are not zero nor is the magnitude of the primer less than or equal to unity at all instances. Initial and final coastal arcs will be included to lower the cost of the transfer. The initial slope is positive and the final slope is negative, which indicates coastal arcs with $dt_0 > 0$ and $dt_f < 0$. Initial and final coastal arcs of 3.3 hours and 3.7 hours, respectively, were found to minimize the cost of the two-impulse transfer. The resulting primer vector magnitude history is presented in Figure 15(b).

Figure 15. Left: The primer vector magnitude history for the reference two-impulse transfer. The transfer is not optimal. The total $\Delta V$ is 99.06 m/s. Right: The primer vector magnitude history for best two-impulse transfer. The transfer is optimal. The total $\Delta V$ is 63.71 m/s.

The inclusion of coastal arcs rendered the transfer optimal, as the primer magnitude slopes are
zero at the terminals and the primer magnitude never exceeds unity. The total cost for this optimal transfer is 63.71 m/s. Two views of the transfer trajectory are shown in Figure 16.

(a) Three-Dimensional View

(b) x-y Planar View

(c) x-z Planar View

Figure 16 Views of the optimal transfer trajectory in the Jupiter-Europa System. The maneuvers are denoted by the yellow dots. The total \( \Delta V \) is 63.71 m/s.

The time of flight for this transfer is just over 16 days. However, the period of the \( L_2 \) halo orbit is only 1.6 days, and the period of the UPO is 4.13 days. The theoretical minimum \( \Delta V \) for this transfer, computed using the method described by Davis et al.,\(^9\) was found to be approximately 27 m/s. It may be possible to lower the transfer cost closer to the theoretical minimum by allowing for longer manifold propagation. Given longer times of flight, the manifolds may come into better alignment, lowering the total cost of the transfer.
CONCLUSIONS

This research has applied primer vector theory to construct locally optimal transfer trajectories between unstable three-body orbits. Locally optimal transfer trajectories were constructed between two Southern halo orbits about the Earth-Moon L$_2$ point. The first transfer constructed did not use invariant manifolds and had a total cost of 113 m/s. In the second transfer, the trajectory asymptotically departed the initial orbit on its unstable manifold and later, asymptotically arrived at the final orbit on its stable manifold. Three mid-course maneuvers were executed to bridge the unstable manifold of the first orbit to the stable manifold of the final orbit. The total cost was only 35.5 m/s. In both cases, primer vector theory was used to determine the locations and times of the impulses. Although both transfers satisfied the conditions for optimality, the transfer that employed invariant manifolds showed a significant decrease in the total cost. The decrease in cost came at the expense of increased transfer time. However, it was shown that the time of flight might be acceptable, or even favorable, for a lunar relay constellation. A third transfer trajectory was constructed in the Earth-Moon system to take a spacecraft from the final orbit back to the initial orbit. This transfer was a reflection of the original transfer about the x-axis and is easily computable due to the inherent symmetry in the CRTBP. Finally, the method was applied to a transfer in the Jupiter-Europa system to connect an L$_2$ halo orbit to a UPO about Europa. This type of transfer may be particularly useful for future Jovian Moon tours.

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