CONNECTING LIBRATION POINT ORBITS OF DIFFERENT ENERGIES USING INVARIANT MANIFOLDS

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This research presents a method of using invariant manifolds to connect libration point orbits. The method presented is applicable to constructing transfers between planar or three-dimensional orbits that have different initial energies. Two deterministic maneuvers are used to connect an unstable manifold trajectory of the first orbit to a stable manifold trajectory of the second orbit. The use of the two-body eccentricity and normalized angular momentum vectors is demonstrated as a viable approach to locating unstable/stable manifold trajectory pairs with low transfer costs. A genetic algorithm is used to vary the parameters that define the transfer. Preliminary results indicate that this method produces fuel costs up to 72% less than transfer trajectories that do not employ the use of manifolds at the expense of increased transfer time. This technique is envisioned as a practical application to decreasing fuel costs and adding flexibility to mission design.

INTRODUCTION

Libration point orbits have been successfully implemented in missions since the 1970s and have the potential to be used in future missions. Halo and Lissajous orbits are planned as the nominal science orbits for several upcoming and proposed science missions, such as the James Webb Space Telescope, XEUS, and Terrestrial Planet Finder. Additionally, many studies have proposed the use of halo orbits for lunar navigation and communication (nav/comm) relay constellations. The orbits proposed for lunar nav/comm relay constellations vary in size, shape, and energy. An inexpensive method of transferring spacecraft between halo orbits would add a great deal of flexibility to such a constellation. Mission designers would have the ability to transfer spacecraft to different orbits to achieve mission objectives or improve coverage characteristics for certain regions of the lunar surface. Furthermore, the connections may be used as part of the constellation, if the connection has desirable coverage characteristics.

The use of halo orbits and their associated invariant manifolds has been proposed for Jovian Moon tours. Many different types of unstable periodic orbits and their associated invariant manifolds exist within the various Jupiter-Jovian Moon three-body systems. Transfers may be constructed between unstable periodic orbits and libration point orbits to form a part of a tour. A Petit Grand Tour of the Jovian system was designed by Koon et al. and Gomez et al. which used invariant manifolds to “leap frog” between moons. The tour they designed began beyond Ganymede in orbit around Jupiter, performed a flyby of Ganymede, transitioned to a temporary capture orbit around Europa, and eventually performed an impulsive maneuver to reach a high inclination orbit around Europa. Recently, Russell located families of unstable periodic orbits around Europa, which

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would have associated invariant manifolds. The final orbit about Europa, chosen for its inclination, may not have the same energy as the manifolds used in the first segments of the tour. The techniques developed here may be used as a mission design tool to decrease the fuel cost of that mission.

Previous research has explored transfers between L\(_1\) and L\(_2\) orbits of the same energy by connecting invariant manifolds to form heteroclinic connections\(^9,11\). These are zero-cost transfers, as a particle asymptotically departs the first orbit on its unstable manifold and asymptotically approaches the second orbit on its corresponding stable manifold. Studies have explored strategies to transfer between orbits of different energy without the use of manifold theory. Howell and Hiday-Johnston developed a method in which they selected departure and arrival states on two halo orbits and connected them using a portion of a Lissajous trajectory\(^12,13\). They employed the use of primer vector theory\(^14\) and extended it to the three-body problem to establish optimal transfers. Gomez et al. used a combination of manifold theory and Floquet theory to construct transfers between halo orbits of different energies\(^15\). They constructed two-maneuver transfers where the first maneuver is performed in the direction tangent to the family containing the halo orbits, and the second maneuver is performed in the direction of the stable manifold. The method presented here is vastly different in nature from the two previously developed strategies. The transfer trajectory constructed in this method involves two deterministic maneuvers to connect the unstable invariant manifold of the first orbit to the stable invariant manifold of the second orbit. Furthermore, techniques from two-body dynamics are employed to determine specific trajectories within the stable and unstable manifolds that produce small transfer costs.

**THE CIRCULAR RESTRICTED THREE-BODY PROBLEM**

The research presented here is conducted within the framework of the Circular Restricted Three-Body Problem (CRTBP). This formulation models the motion of a particle of negligible mass (i.e. a spacecraft) under the influence of two larger bodies, the primary and the secondary. The primary and the secondary are collectively termed primaries. The primaries are assumed to rotate in circular orbits about the center of mass of the system, otherwise known as the barycenter. The reference frame rotates about the barycenter at the same rotation rate as the two primaries. The \(x\)-axis extends from the origin through the secondary, the \(z\)-axis extends in the direction of the angular momentum of the system, and the \(y\)-axis completes the right-hand coordinate frame. The equations describing the motion of the third body may be written as

\[
\begin{align*}
\ddot{x} &= 2\dot{y} + x - (1 - \mu) \frac{x + \mu}{r^3_1} - \mu \left( \frac{x - 1 + \mu}{r^3_2} \right) \\
\ddot{y} &= -2\dot{x} + y - (1 - \mu) \frac{y}{r^3_1} - \mu \frac{y}{r^3_2} \\
\ddot{z} &= - (1 - \mu) \frac{z}{r^3_1} - \mu \frac{z}{r^3_2},
\end{align*}
\]  

(1)

where \(r_1\) and \(r_2\) are equal to the distance from the third body to the primary and secondary, respectively:

\[
\begin{align*}
r^2_1 &= (x + \mu)^2 + y^2 + z^2 \\
r^2_2 &= (x - 1 + \mu)^2 + y^2 + z^2,
\end{align*}
\]  

(2)

and \(\mu\) is the mass parameter used to non-dimensionalize the system. The reader is directed to Szebehely\(^16\) for a derivation of the equations of motion.
The dynamics of the CRTBP allow the existence of an integral of motion to exist in the rotating frame. If the equations of motion (Equation 4) are multiplied by $2\dot{x}$, $2\dot{y}$, and $2\dot{z}$, respectively, summed together, and integrated, the integral of motion known as the Jacobi constant emerges:

$$C = 2\Omega - V^2,$$  

where

$$\Omega = \frac{1}{2} (x^2 + y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}$$

$$V^2 = (x^2 + y^2 + z^2).$$

The Jacobi constant is analogous to energy in the two-body problem, in that the Jacobi constant cannot change unless the spacecraft is perturbed by something other than the two primaries. Note that it is only a function of the position and velocity magnitude expressed in the rotating frame.

**Periodic orbits**

The equations of motion of the CRTBP permit the existence of five equilibrium points, also known as the libration, or Lagrange points. Families of periodic and quasi-periodic orbits exist about the libration points and have been studied by a host of researchers, such as Broucke and Henon. A Lissajous orbit is a quasi-periodic orbital trajectory that winds around a torus, but never closes in on itself. Halo orbits are a special case of Lissajous orbits where the in plane and out of plane frequencies are equal. Halo orbits are periodic and are aptly named: they appear to form a halo about the Moon when viewed from the Earth. Halo orbits are three-dimensional; a two-dimensional halo orbit is referred to as a Lyapunov orbit. Several methods exist for numerically computing libration point orbits. The orbits presented here were computed using two methods: a Richardson-Cary expansion and a Single-Shooting Algorithm.

**The parameter $\tau$** For periodic orbits in the CRTBP, it is useful to introduce the parameter $\tau$, which is used to describe the position of a particle on a periodic orbit. This is similar to the mean anomaly in the two-body problem, which is related to time and not a physical angular measurement. The value of $\tau$ ranges from 0–2$\pi$ radians, or 0°–360°, increasing in the direction of orbital motion. $\tau$ is defined to be zero at the orbit’s initial $x$-axis crossing in the $+v_y$ direction, and 180° at the $-v_y$ crossing, as shown in Figure 1.

**Stability of halo orbits** A stability index, $\nu$, can be computed for each unstable periodic orbit, based on $\lambda_{max}$, the largest eigenvalue from the monodromy matrix. The monodromy matrix is defined as the state transition matrix propagated to the end of one orbital period.

$$\nu = \frac{1}{2} \left( \frac{1}{\lambda_{max}} + \frac{1}{\lambda_{max}} \right).$$  

A stable orbit has a stability index of one, and stability indices greater than one correspond to unstable orbits. As the stability index increases, the orbital stability decreases.

**IN Variant MANIFOLDS**

Libration point orbits are generally unstable. Each unstable periodic orbit will have at least one stable and one unstable eigenvalue with corresponding eigenvectors. If a particle on an unstable periodic orbit is given a perturbation in the unstable direction, it will exponentially fall away from
the nominal orbit, tracing out a smooth trajectory away from the orbit. Conversely, a particle given the right initial conditions will follow a smooth trajectory that exponentially approaches the unstable orbit and eventually arrives on that orbit from the orbit’s stable direction. The terms for the full set of exponentially diverging and converging trajectories are the unstable and stable invariant manifolds of the orbit. The term invariant simply means that through the propagation of time, a point on the manifold will stay on the manifold.

The unstable manifold \((W^U)\) includes the set of all possible trajectories that a particle on a nominal orbit could traverse if it was perturbed in the direction of the orbit’s unstable eigenvector. The stable manifold \((W^S)\) includes the set of all possible trajectories that a particle could take to arrive onto the nominal orbit. The manifolds form tube-like structures for periodic orbits. Each orbit has two associated stable and unstable manifold sets: one corresponding to a positive perturbation, and one corresponding to a negative perturbation. Planar views of the unstable and stable manifolds of a halo orbit are shown in Figure 2. This paper will follow a common representation for the invariant manifolds: the unstable manifold trajectories will be red, and the stable manifold trajectories will be green. The parameter \(\tau\) is used to define the location where a perturbation is given to create a manifold trajectory. The stability index dictates how quickly the manifolds will depart the orbit for a given perturbation value. Larger stability indices correspond to manifolds that will depart from their nominal orbit more quickly than the manifolds of orbits with smaller stability indices.

**Using invariant manifolds for transfers between orbits**

The trajectories within the invariant manifold of an unstable periodic orbit often traverse a wide range of locations within the three-body system, and the idea to use them to connect orbits within the restricted three-body problem is not a new one. Conley\(^{22}\) and McGehee\(^{23}\) proved the existence of homoclinic orbits, orbits that are both forward and backward asymptotic to an unstable periodic orbit. From a dynamical systems standpoint, it may be stated that a homoclinic orbit occurs when the stable and unstable manifolds of a certain equilibrium point or periodic orbit are identical. From a mission design point of view, a spacecraft could depart on a trajectory contained within
the unstable manifold of an orbit and arrive back to the orbit at a later time on the stable manifold without performing any deterministic maneuvers. In a similar sense, a trajectory within the unstable manifold of one orbit may depart that orbit and eventually approach a second orbit along its stable manifold. The trajectory is thus contained within both the unstable manifold of the first orbit and the stable manifold of the second orbit. Such a trajectory forms what is known as a heteroclinic connection between the two unstable orbits. Heteroclinic connections were numerically proven to exist by Koon et al. and Gomez et al.\(^9,24\) Homoclinic orbits and heteroclinic connections between three-body orbits have been used to explain transport mechanisms for spacecraft and comets. Using dynamical systems theory, Lo and Ross\(^{25}\) noted that the orbit of the comet \textit{Oterma} appeared to shadow the invariant manifolds of the libration points \(L_1\) and \(L_2\) in the Sun-Jupiter frame, and later, Koon et al. showed that the path of the comet closely followed a homoclinic-heteroclinic chain. Koon et al. also explored the numerical construction of orbits with prescribed itineraries to describe the resonant transitions exhibited by the comet \textit{Oterma}. Lo and Parker\(^{26}\) explored the use of invariant manifolds to chain together periodic three-body orbits. Many of the techniques employed in the previous research have been used here. Homoclinic orbits and heteroclinic connections lay the foundations for a more general case of transfers between unstable orbits in the CRTBP. These maneuver-free transfers only exist between orbits with the same energy, or Jacobi constant. Obviously, if a transfer is desired between orbits of different energies, at least one maneuver will be required to complete the transfer.

**POINCARÉ MAPS**

Poincaré maps have been successfully used to locate free transfers between orbits of the same energy in the CRTBP. A Poincaré map replaces the flow of an \(n^{th}\) order system with a discrete-time system with the order of \((n - 1)\). For a given \(n\) dimensional system, \(\dot{x} = f(x)\), an \((n - 1)\) dimensional surface of section \(\Sigma\) is placed transverse to the flow. A Poincaré map is created by following trajectories and mapping the position and/or time coordinates at each instance that the trajectory has an intersection with the surface \(\Sigma\).
Consider a simple transfer between planar Lyapunov orbits, one at L_1 and one at L_2. Each of these orbits has a set of stable and unstable invariant manifolds. In the planar CRTBP, each point along a manifold may be characterized by a four-dimensional state \([x, y, \dot{x}, \dot{y}]\). If a Poincaré section is placed in \(\mathbb{R}^4\) at some \(x\)-position, the resulting intersection is a surface in \(\mathbb{R}^3\). If the two orbits have the same Jacobi constant, then each point along any trajectory within both orbits’ manifolds will have the same Jacobi constant, and the phase space of the problem is reduced to \(\mathbb{R}^2\). When integrating the invariant manifolds to the surface of section, the manifolds of both orbits appear as curves in the two-dimensional Poincaré map. Any intersections of these curves correspond to free transfers between the two orbits.

The left side of Figure 3 illustrates the unstable and stable manifolds integrated to the first intersection with the surface of section. The positions and velocities in the \(y\)-direction of both manifolds at the surface of section are plotted on the right side of Figure 3. One can see that there are two intersections of the manifold curves in the Poincaré map that correspond to the two free transfers indicated in the figure.

![Figure 3](image-url) An illustration of the process of using a Poincaré map to identify free transfers between two Lyapunov orbits in the Earth-Moon system. Both orbits have a Jacobi constant of 3.13443929. Left: the unstable manifold of an L_1 Lyapunov orbit and the stable manifold of an L_2 Lyapunov orbit integrated to the surface of section. Right: the corresponding Poincaré map and two free transfers.

It should be noted that although the transfer is termed “free,” in reality, very small maneuvers are required to transition from the initial orbit onto the unstable manifold and to transition from the stable manifold onto the second orbit. If this theoretical orbit was constructed using the full ephemeris, the two maneuvers would be on the order of a few meters per second or less.

More complicated Poincaré maps are required to locate transfers between three-dimensional orbits of the same energy. The reader is directed to the work of Gomez et al.\(^9,11\) and Koon et al.\(^24\) for examples. If the initial and final orbits do not have the same energy, the system can no longer be reduced in dimension. However, invariant manifolds may still be used to construct transfers, although at least one maneuver will be required to complete a transfer connecting the unstable manifold of the first orbit to the stable manifold of the second orbit.
The concept of a bounding sphere is introduced to take the place of the planar Poincaré section, \( \Sigma \). A sphere of radius \( R \) is placed around the center of mass of one of the two primaries. Trajectories on the invariant manifolds are integrated, and the state is stored each time a trajectory pierces the sphere, as illustrated in Figure 4. Each trajectory piercing is then integrated inside the bounding sphere such that the position differences of successive integration points are nearly constant, as shown in Figure 5. A searching algorithm is used to determine the stable and unstable manifold pair that will produce the smallest total \( \Delta V \). The method used to determine the unstable/stable manifold combination and the maneuver locations that produce the smallest total \( \Delta V \) is the primary concern of this paper and will be discussed in upcoming sections.

In this research, the bounding sphere will be placed about the secondary. This is due to several factors: first, in any Sun-planet system, transfers that occur near the Sun are not feasible due to extreme environmental conditions. This may also be true for planet-moon systems, such as the Jupiter-Europa system. Second, following invariant manifolds from an orbit at \( L_1 \) or \( L_2 \) to the vicinity of the larger primary may take a considerable amount of time - longer than feasible for mission design purposes. Thus, for practical reasons, the bounding sphere will be centered on the secondary.

**Two-body dynamics within the bounding sphere** If the bounding sphere’s radius is less than the radius of the sphere of influence of the secondary, the trajectories within the bounding sphere will be mainly influenced by the secondary. This allows techniques from two-body dynamics to be used to help determine the minimum transfer cost. In order to ensure that the motion of the particle is dominated by the secondary for the two-body analysis to be valid, the bounding sphere radius is selected to be approximately one-third the radius of the sphere of influence (e.g. about 22,000 km in the Earth-Moon system).
Figure 5  The manifold trajectories that pass through the sphere are highlighted (left) and each trajectory is integrated such that successive points are spaced approximately equal in position (right).

It is possible to capture the shape and orientation of a trajectory by analyzing two parameters associated with the two-body problem, the normalized angular momentum vector, $\overrightarrow{h}_{\text{norm}}$, and the eccentricity vector, $\overrightarrow{e}$. The vectors $\overrightarrow{h}_{\text{norm}}$ and $\overrightarrow{e}$ are computed from the position and velocity with respect to the secondary, $\overrightarrow{r}_{\text{sec}}$ and $\overrightarrow{v}_{\text{sec}}$, and gravitational parameter of the secondary, $\mu_{\text{sec}}$ (not to be confused with $\mu$, the parameter used to non-dimensionalize the three-body system).

$$\overrightarrow{h}_{\text{norm}} = \frac{\overrightarrow{r}_{\text{sec}} \times \overrightarrow{v}_{\text{sec}}}{\sqrt{\mu_{\text{sec}}}}$$  \hspace{1cm} (8)$$

$$\overrightarrow{e} = \frac{\overrightarrow{v}_{\text{sec}} \times (\overrightarrow{r}_{\text{sec}} \times \overrightarrow{v}_{\text{sec}})}{\mu_{\text{sec}}} - \frac{\overrightarrow{r}_{\text{sec}}}{r_{\text{sec}}}$$  \hspace{1cm} (9)$$

The vectors $\overrightarrow{h}_{\text{norm}}$ and $\overrightarrow{e}$ are independent of the periapsis epoch and semimajor axis (only used for normalization), but are functions of the other four orbital elements ($e, i, \omega, \Omega$).

Each point along a manifold trajectory within the bounding sphere may be expressed in terms of these two vectors. In a two-body dynamics sense, minimizing the differences in the normalized angular momentum and eccentricity vectors between two points minimizes the differences in the shape and orientation of the respective orbits that contain the two points. Thus, these vectors will be used to select unstable and stable manifold trajectories that closely match in shape and orientation. Let us define the differences of the vectors between a point on the unstable manifold and a point on the stable manifold as:

$$\Delta h_{\text{norm}} = |\overrightarrow{h}_{\text{norm,}S} - \overrightarrow{h}_{\text{norm,}U}|$$  \hspace{1cm} (10)$$

$$\Delta e = |\overrightarrow{e}_{S} - \overrightarrow{e}_{U}|,$$  \hspace{1cm} (11)$$

where the subscripts U and S denote the unstable and stable manifolds, respectively.

Now, let $\kappa$ be defined as the sum of the differences of the $\overrightarrow{h}_{\text{norm}}$ and $\overrightarrow{e}$ vectors:

$$\kappa = \Delta h_{\text{norm}} + \Delta e.$$  \hspace{1cm} (12)$$
The parameter $\kappa$ can now be used to compare the two body parameters of the unstable and stable manifold trajectories. We will numerically demonstrate that within the bounding sphere, the stable/unstable manifold trajectory pair which minimizes $\kappa$ will generally produce the transfer with the lowest fuel cost.

**CONSTRUCTING A TRANSFER BETWEEN TWO ORBITS**

This section provides a description of the methodology used to construct a transfer between the unstable manifold of the first orbit and the stable manifold of the second orbit. First, the unstable manifold points of the first orbit and the stable manifold points of the second orbit are computed within the bounding sphere, as depicted in Figures 4 and 5. Next, stable and unstable points with small $\kappa$ values are located. The locations where the stable and unstable manifold points have small values of $\kappa$ also match closely in position and velocity. This is termed a close approach. An example close approach is shown in Figure 6.

Two maneuvers will be used to construct a trajectory. The first maneuver will be performed on the unstable manifold at some time $\Delta t_1$ before the close approach. The first maneuver will target the position on the stable manifold at some time $\Delta t_2$ after the close approach. The total duration of the transfer trajectory between the manifolds is some $\Delta t$, which may or may not be equal to $\Delta t_1 + \Delta t_2$. An optimization method is used to determine the optimal time between the maneuvers, so as to minimize the total $\Delta V$. The optimization process is described in the following section. The magnitude and direction of the first maneuver, $\Delta V_1$, are computed using Level 1 of a Differential Corrector. Once this maneuver is computed, the transfer trajectory is propagated forward in time after $\Delta V_1$ to the intersection with the stable manifold. Here, the second maneuver, $\Delta V_2$, is executed to match the transfer trajectory’s velocity to the velocity on the stable manifold. Note that the times $\Delta t_1$ and $\Delta t_2$ may or may not be equal. The process is illustrated in Figures 7 and 8.
Figure 7. The maneuver locations on the unstable and stable manifolds.

Figure 8  Left: The first maneuver on the unstable manifold, denoted by the pink circle, targets a state on the stable manifold. The second maneuver, denoted by the light green circle, corrects the velocity at the end of the bridging trajectory (black) to match the velocity on the stable manifold. Right: The final trajectory connecting the unstable manifold of the first orbit to the stable manifold of the second orbit is continuous in position and requires two impulsive maneuvers.

Optimizing the $\Delta V$ between two points

In the two-maneuver transfer described above, there are five parameters that define the cost of the transfer between two orbits: the starting point on the initial orbit that defines the unstable manifold, $\tau_U$, the time to propagate the unstable manifold, $t_U$, the ending state on the final orbit that defines the stable manifold, $\tau_S$, the time to propagate the stable manifold, $t_S$, and the time between the
maneuvers, $\Delta t$. It is possible to determine the transfer time, $\Delta t$, that will minimize the total transfer $\Delta V$ required, given the two states where the maneuvers are performed. This will reduce the number of parameters that define a transfer from five to four. A cost function, $J$, is defined in terms of the two maneuvers, such that

$$J = |\Delta V_1| + |\Delta V_2|, \quad (13)$$

where

$$|\Delta V_i| = \sqrt{\Delta V_i \cdot \Delta V_i}. \quad (14)$$

Let $\Delta t$ be equal to the time between the two maneuvers. Take the partials of the cost function, $J$, with respect to $\Delta t$ to obtain the following equation:

$$\frac{\partial J}{\partial \Delta t} = \frac{\partial |\Delta V_1|}{\partial \Delta V_1} \cdot \frac{\partial \Delta V_1}{\partial \Delta t} + \frac{\partial |\Delta V_2|}{\partial \Delta V_2} \cdot \frac{\partial \Delta V_2}{\partial \Delta t}. \quad (15)$$

Realizing that

$$\frac{\partial |\Delta V_i|}{\partial \Delta V_i} = \frac{1}{2} \cdot \frac{1}{\sqrt{\Delta V_i \cdot \Delta V_i}} \cdot (2\Delta V_i) = \frac{\Delta V_i}{|\Delta V_i|}, \quad (16)$$

Equation 16 may be rewritten as

$$\frac{\partial J}{\partial \Delta t} = \frac{\Delta V_1}{|\Delta V_1|} \cdot \frac{\partial \Delta V_1}{\partial t} + \frac{\Delta V_2}{|\Delta V_2|} \cdot \frac{\partial \Delta V_2}{\partial t}. \quad (17)$$

Minimizing Equation 18 will minimize the total $\Delta V$ for the two-maneuver transfer. Stated another way, minimizing Equation 18 will find the value of $\Delta t$ that minimizes the sum of the maneuvers that are required complete the transfer. It is now necessary to solve for the unknown terms in Equation 18.

Recall that the first maneuver is executed on the unstable manifold to target some state on the stable manifold. The position on the stable manifold, $r_S$, may be expressed in terms of the following variables:

$$r_S = \varphi_r \left( t_U + \Delta t, \quad r_U, \quad v_U + \Delta V_1, \quad t_U \right), \quad (18)$$

where $r$ denotes position, $v$ denotes velocity, and the subscripts $U$ and $S$ denote the unstable and stable manifolds respectively. Let the velocity after the execution of first maneuver be $\overline{\Delta V}_0$ defined as

$$\overline{\Delta V}_0 = r_U + \Delta V_1. \quad (19)$$

Take the partials of Equation 19 with respect to $\Delta t$, to obtain the following:

$$\frac{\partial r_s}{\partial \Delta t} = \left. \frac{\partial \varphi_r}{\partial \Delta t} \right|_{t_U + \Delta t} + 0 + \frac{\partial \varphi_r}{\partial r_U} \cdot \frac{\partial \overline{\Delta V}_0}{\partial \Delta t}, \quad (20)$$
The position on the stable manifold and the velocity on the unstable manifold before the first maneuver remain constant, despite variations to $\Delta t$. After simplifying and rearranging Equation 21, the variations of $\Delta V_1$ with respect to changes in the transfer time are found:

$$0 = \phi_v (t_u + \Delta t) + \Phi_{rv} \cdot \frac{\partial V_1}{\partial \Delta t}$$

$$\frac{\partial \Delta V_1}{\partial \Delta t} = -\Phi_{rv}^{-1} \phi_v (t_u + \Delta t),$$

where $\Phi$ is the state transition matrix, integrated forward $\Delta t$ from the state after the execution of $\Delta V_1$. If the 6x6 $\Phi$ matrix is partitioned into 4 submatrices, $\Phi_{rv}$ is upper right 3x3 submatrix:

$$\left[ \begin{array}{cc} \Phi_{rr} & \Phi_{rv} \\ \Phi_{vr} & \Phi_{vv} \end{array} \right].$$

Note that $\phi_v (t_u + \Delta t)$ is the velocity of the state at time $t_u + \Delta t$, propagated from the position on the unstable manifold after the execution of $\Delta V_1$. In other words, $\phi_v (t_u + \Delta t)$ is the velocity immediately before the execution of $\Delta V_2$.

The second maneuver, $\Delta V_2$, may be written in terms of the following variables:

$$\Delta V_2 = \tau_s - \phi_v (t_u + \Delta t, \ tau, \ \tau_u + \Delta V_1, \ t_u).$$

Taking the partial derivative of Equation 23 with respect to $\Delta t$, simplifying, and rearranging, an expression for the variations of $\Delta V_2$ with respect to changes in the transfer time is found:

$$\frac{\partial \Delta V_2}{\partial \Delta t} = \frac{\partial \Delta V_s}{\partial \Delta t} - \left[ \frac{\partial \phi_v}{\partial \Delta t} \bigg|_{t_u+\Delta t} + \frac{\partial \phi_v}{\partial \Delta V_0} \cdot \frac{\partial \Delta V_1}{\partial \Delta t} \right]$$

$$\frac{\partial \Delta V_2}{\partial \Delta t} = 0 - \left[ \frac{\partial \phi_v}{\partial \Delta t} \bigg|_{t_u+\Delta t} + \Phi_{vv} \cdot \frac{\partial \Delta V_1}{\partial \Delta t} \right]$$

$$\frac{\partial \Delta V_2}{\partial \Delta t} = -\left[ \phi_a (t_u + \Delta t) + \Phi_{vv} \cdot \frac{\partial \Delta V_1}{\partial \Delta t} \right],$$

where the subscript $a$ denotes the acceleration. The term $\Phi_{vv}$ is the lower right submatrix of the partitioned state transition matrix. The partials of $\Delta V_1$ and $\Delta V_2$ with respect to time, given by Equations 22 and 24, can be substituted back into Equation 18. Then, the transfer time that minimizes Equation 18 can be quickly computed by an iterative secant method process. In this fashion, the transfer time is found such that the total $\Delta V$ required for the transfer will be minimized for the maneuver locations selected.

**Two-body parameters and their correlation to total transfer cost**

Using all of the techniques described above, the relationship between the two-body parameter $\kappa$ and the total transfer cost was determined. Unstable and stable manifolds were propagated to bounding sphere intersections, and points on each manifold were computed within the bounding
sphere. A value of $\kappa$ was computed for each combination of stable and unstable manifold points whose position differences were within a specified tolerance. The position difference criterion was added to reduce the number of $\kappa$ computations. It was noticed that within the bounding sphere, the points with the smallest $\kappa$ parameters matched closely in position. Next, the total $\Delta V$ was computed to complete the transfer between the manifolds. The maneuver locations along each manifold trajectory were varied such that the duration between the maneuvers was nearly constant. Figure 9 illustrates how changes to the parameter $\kappa$ affect the total $\Delta V$ of the transfer.

Figure 9 shows an approximate linear increase between $\kappa$ and total transfer $\Delta V$. As the two-body parameters of an unstable manifold more closely match the two-body parameters of a stable manifold, the total $\Delta V$ required to complete the transfer decreases. Thus, within the bounding sphere, stable/unstable manifold trajectory combinations with small values of $\kappa$ should produce small $\Delta V$ costs. Recall that each stable and unstable manifold can be classified by their respective $\tau$ values, so the values $\tau^U$ and $\tau^S$ are used to parameterize the problem.

**Genetic algorithms**

A genetic algorithm (GA) is a numerical searching technique used to find exact or approximate solutions to optimization and search problems. GAs employ Survival of the Fittest theory to create populations of solutions that evolve toward better solutions. The algorithm begins with a population of randomly generated solutions to the problem and proceeds by creating new generations of solutions. New generations are created by evaluating the fitness of every solution in the population and stochastically selecting the best solutions in the current population based on their fitness values and modifying them by recombination and/or random mutation to form a new population. The new population is then used in the next iteration of the algorithm. This process continues until a satisfactory solution has been located, or the maximum number of generations has been produced.

A GA was implemented in this study to locate minimum transfer costs. The GA iterated over four parameters, the initial and final $\tau$ values and the times to propagate the unstable and stable manifolds. An initial population of solutions was randomly generated for initial and final $\tau$ com-
binations which had the minimum values for \( \kappa \). The propagation times along the manifolds were selected randomly, and the time between the maneuvers was computed from the previously described optimization equations. The total \( \Delta V \) to complete the transfer was used as the measure of fitness for a good trajectory. Generations consisting of one thousand solutions were computed, and the 25\% with the best fitness were used to create subsequent generations. Typically, less than 10 generations were necessary for a convergent solution.

**Theoretical minimum transfer \( \Delta V \)s**

A theoretical minimum transfer cost can be found for each trajectory, under specified conditions. Sweetser derived the following equation to compute a \( \Delta V \) for an Earth-Moon transfer based on changes in the Jacobi constant:

\[
|\Delta C| = 2v|\Delta V| + |\Delta V|^2,
\]

(24)

where \( v \) is the velocity in the rotating frame at the location where the maneuver is performed.\(^{28}\)

Rearrange Equation 25 into a simple quadratic in terms of \( |\Delta V| \), and it can be seen that for a given \( \Delta C \), the rotational velocity, \( v \), should be maximized to get the minimum \( \Delta V \). The maximum rotational velocity occurs when the potential, \( \Omega \), is maximized (See Equation 5). For a given three-body system, there are three regions of high potential: far away from the barycenter (in the \( x-y \) plane), near the primary, and near the secondary. As the transfers in this study perform flybys of the secondary, the potential will be calculated based on locations near the secondary. The potential increases as the distance to the center of the secondary decreases. However, it is obviously not feasible to perform maneuvers at locations very close to the surface of the secondary due to planetary quarantine issues or possible collisions. Thus, the highest potential can occur at some boundary condition defined as an altitude above the secondary. In this study, an altitude of 1000 km above the surface of the Earth is used. In the Sun-Earth system, the potential is nearly identical at all locations along a sphere of this altitude. Thus, the specific \( x, y, \) and \( z \)-coordinates make no difference for the potential calculation, so long as the radius is at a 1000 km altitude from the surface.

To compute the minimum theoretical transfer \( \Delta V \), certain ideal conditions are assumed. The minimum-cost transfer trajectory would follow the unstable invariant manifold of the first orbit to a perfect position intersection with the stable manifold of the second orbit, so only one maneuver would need to be performed. In this ideal case, at this position intersection of the manifolds, the difference in the velocities between the manifolds would also be at a minimum. The position intersection of the manifolds would occur at one of the boundary conditions, so as to maximize the potential. At this boundary location, a maneuver would be performed to correct the velocity and change the Jacobi constant. The maneuver magnitude is computed by solving Equation 25, using the change in Jacobi constants between the orbits, and the magnitude of the rotational velocity at the boundary condition. This represents an estimate for the ideal, theoretical minimum \( \Delta V \) to transfer between the orbits.

Each transfer trajectory will have a minimum flyby altitude above the secondary, which will be greater than or equal to the minimum allowable altitude of 1000 km, but lower than the radius of the bounding sphere. The minimum radius of the transfer trajectory itself will be used to compute a second theoretical minimum \( \Delta V \). This represents the theoretical minimum for the particular trajectory under analysis, as it is unlikely that the manifolds have similar \( \kappa \) parameters at the minimum altitude where the potential is maximized.
NUMERICAL RESULTS

This method was used to compute transfers between two L₁ halo orbits in the Sun-Earth/Moon-barycenter system. The halo orbits were characterized by the z-amplitudes. The initial z-amplitude varied from 160,000 km to 240,000 km while the z-amplitude of the final halo orbit was held constant at 110,000 km. Howell and Hiday-Johnston investigated transfers between these specific orbits using an application of primer vector theory.¹² Their results will be compared to the ones presented here, in terms of both total fuel expenditure and transfer time.

A planar view of the transfer constructed from the halo orbit with an initial z-amplitude of 160,000 km is shown in Figure 10(a). The motion in the z-direction is relatively small compared to the motion in the x- and y-directions. The transfer departs the initial halo orbit, following its unstable manifold, denoted by the red line in Figure 10(a). After this coasting phase of the transfer, a maneuver is executed to bridge the unstable manifold to the stable manifold. After the execution of the first maneuver, the spacecraft coasts along the black portion of the trajectory to a position intersection with the stable manifold where the second maneuver is performed. The spacecraft then follows the green stable manifold to the final orbit.

Interestingly, a solution was located which follows the middle portion of the trajectory twice. This transfer is shown in Figure 10(b) and is denoted as a two-loop transfer. The idea that three-body trajectories can retrace their paths has been explored by Lo and Parker in their work on chaining periodic orbits.²⁶ This implies that the orbital trajectory shown here could theoretically retrace its path an infinite number of times. However, the time constraints on mission designs, and the likely need for large amounts of station-keeping renders such an idea as practically unfeasible.

The transfers constructed for all other initial z-amplitudes also had both one-loop and two-loop solutions. Additionally, the transfers followed very similar trajectory paths. The transfer trajectory projections into the x-y plane look nearly identical for all of the constructed orbit transfers. The motion in the z-direction increased as the initial z-amplitude of the orbit increased.

Figure 11 presents the total ∆V costs for all transfer trajectories. The one- and two-loop transfers
are shown, as well as the costs for the solutions computed by Howell and Hiday-Johnston and the theoretical minimum transfer costs.

![Figure 11. Total transfer cost comparison.](image)

The one-loop and two-loop transfers show a drastic decrease in the total cost to transfer between the two orbits compared to the costs previously computed. The transfer costs decreased by as much as 36% for the one-loop case and 72% for the two-loop case. The one-loop transfer results are nearly linear, as are the results from Howell and Hiday-Johnston. Gomez stated that the cost of the transfer between orbits should be nearly proportional to the z-amplitude difference between the orbits. This seems intuitive, because the energy difference between the two halo orbits increases as their z-amplitude differences increases. However, this may not necessarily hold true for transfers constructed using invariant manifolds. Consider Case A, the transfer with the initial z-amplitude of 160,000 km and Case B, the transfer with the initial z-amplitude of 180,000 km. Although the energy difference is higher in Case B, the transfer cost is smaller. A quick investigation of the \( \kappa \) values showed that the \( \kappa \) parameter was almost 50% lower for Case B than for the Case A. Consequently, the transfer for Case B comes within 3.2 m/s of the theoretical minimum transfer \( \Delta V \) for the two orbits.

Table 1 presents a summary of the total transfer costs for one- and two-loop transfers, as well as the cost of the solutions generated by Howell and Hiday-Johnston. The theoretical minimum \( \Delta V_s \) for the transfers are computed, using an altitude of 1000 km above the surface of the secondary. The final column shows the minimum transfer cost for the specific trajectory, based on the minimum distance that the transfer passes from the center of the secondary. The trajectory minimum is nearly identical for the one-loop and two-loop transfers. The two-loop transfer retraces the path of the one-loop transfer, so the minimum distance to the center of the secondary is nearly the same. The minimum flyby altitude for all transfer cases was approximately 7500 km.

Table 2 presents the times of flight necessary for each transfer trajectory. The clear drawback to the decrease in the fuel cost of the transfers using invariant manifolds is a seven-fold and eleven-fold increase in total time of flight for the one-loop and two-loop transfers, respectively. This suggests that the methods presented here would be more applicable in a three-body system with smaller time scales, such as the Earth-Moon system, or the Jupiter-Europa system.
<table>
<thead>
<tr>
<th>Initial $z$-Amplitude</th>
<th>Howell Solution</th>
<th>Davis 1-Loop</th>
<th>Davis 2-Loop</th>
<th>Theoretical Minimum</th>
<th>Trajectory Minimum</th>
</tr>
</thead>
<tbody>
<tr>
<td>160,000 km</td>
<td>26.36</td>
<td>20.20</td>
<td>10.17</td>
<td>0.07</td>
<td>0.28</td>
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<td>180,000 km</td>
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<td>23.23</td>
<td>3.37</td>
<td>0.11</td>
<td>0.42</td>
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<td>200,000 km</td>
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<td>28.12</td>
<td>12.26</td>
<td>0.15</td>
<td>0.56</td>
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<tr>
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<td>–</td>
<td>34.79</td>
<td>24.15</td>
<td>0.19</td>
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<td>240,000 km</td>
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<td>43.40</td>
<td>36.89</td>
<td>0.24</td>
<td>0.88</td>
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</table>

<table>
<thead>
<tr>
<th>Initial $z$-Amplitude</th>
<th>Howell Solution</th>
<th>Davis 1-Loop</th>
<th>Davis 2-Loop</th>
<th>Davis 1-Loop</th>
<th>Davis 2-Loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>160,000 km</td>
<td>129.13</td>
<td>863.35</td>
<td>1355.83</td>
<td>1355.83</td>
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<tr>
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<td>865.39</td>
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<tr>
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<td>873.72</td>
<td>1364.90</td>
<td>1364.90</td>
<td>1364.90</td>
</tr>
</tbody>
</table>

**TRANSFER IN THE EARTH-MOON SYSTEM**

A transfer was constructed between two orbits in the Earth-Moon system to demonstrate the method in a different three-body system. The initial orbit is a southern halo orbit about L$_2$ with a Jacobi constant of 3.0327 and a period of 7.5 days. The final halo orbit is also about L$_2$, and has a Jacobi constant of 3.06 and a period of 13.6 days. The initial orbit was designed by Hamera et al. for use in a Lunar nav/comm relay and was optimized for south pole coverage. The final orbit was designed by Hill et al. for a lunar nav/comm and gravity mission and has excellent far side coverage. The purpose of transferring between two such orbits may be to move assets in a relay constellation that are focused on south pole coverage to an orbit which provides better far side lunar coverage. The ability to inexpensively transfer between two such orbits adds flexibility to a lunar nav/comm constellation.

Figure 12 presents two views of a transfer constructed in the Earth-Moon system. The majority of the trajectory is spent asymptotically departing the first orbit on its unstable manifold. It can be seen in Figure 12(b) that the trajectory closely follows the path of the nominal orbit for almost four revolutions. This is due to the fact that the initial orbit has a very low stability index and departs the nominal orbit relatively slowly. The total $\Delta V$ for this transfer is 35 m/s. The transfer had a theoretical minimum $\Delta V$ of 7.55 m/s, given that its minimum altitude above the lunar surface was 998 km. The time of flight for this transfer is 104 days. However, as seen from Figure 12, the trajectory stays on the far side of the Moon, spending the majority of its time of flight in the vicinity of the initial orbit. This path provides excellent coverage of the south pole and far side of the Moon during the duration of transfer. Thus, the transfer itself can be used as part of the relay constellation.
CONCLUSIONS

This study has demonstrated a method for constructing transfers between halo orbits of different energies using invariant manifolds. The transfer once defined by five parameters has been reduced to two parameters, the maneuver locations. The time between the maneuvers can be optimized using a cost function and knowledge of the state transition matrix along the trajectory connecting the invariant manifolds. It was numerically demonstrated that the two-body parameter $\kappa$ can be used to analyze invariant manifold trajectories within the sphere of influence of the secondary. As the two body parameters of an unstable manifold more closely matched the two-body parameters of a stable manifold, the total $\Delta V$ to complete the transfer decreased. Using this knowledge, potential starting points of the transfer on the initial orbit and potential ending points of the transfer on the final orbit were determined by analyzing the parameter $\kappa$.

The transfers constructed using this method produce total transfer costs up to 72% less than the cost of transfers that do not employ the use of invariant manifolds. The decrease in total $\Delta V$ was accompanied by a seven- to eleven-fold increase in the total transfer time for the trajectories analyzed in the Sun-Earth system.

A transfer was constructed in the Earth-Moon system between two $L_2$ halo orbits. The constructed transfer was envisioned as a part of a lunar nav/comm relay constellation. Despite taking 104 days to reach the final orbit, the trajectory spent the majority of its time on the far side of the Moon, asymptotically departing the first orbit, and asymptotically arriving at the destination orbit. A spacecraft on that transfer trajectory would retain its excellent coverage characteristics of the lunar south pole and far side, and therefore, the whole duration of the transfer would be useful to the mission.
This method is best suited for three-body systems with smaller time constants. The two-loop transfers in the Sun-Earth system had similar times of flight, in non-dimensional units, to the transfer in the Earth-Moon system. However, since the time constant is much smaller in the Earth-Moon system, the transfer durations can be practical for mission design. Thus, this method would work well in three-body systems such as any of the Jupiter-Jovian Moon or Saturn-Saturnian Moon systems.

**FUTURE WORK**

Future work will focus on developing methods to determine the maneuver locations on the respective manifolds. If the maneuvers are performed within the radius of the bounding sphere, two-body parameters may be used to compute maneuver locations. It is surmised that this problem may be likened to a two-body Hohmann transfer, in which the two burns are tangential. In theory, the maneuvers will be performed at locations on the manifolds where the burn is as close to tangential as possible. However, the best place to perform the maneuvers may not be within the bounding sphere. It may also be possible to apply primer vector theory to the problem to determine optimal maneuver locations, as done by Howell and Hiday-Johnston.

Subsequent research will also seek to implement this method for practical missions. The Petit Grand Tour of the Jovian system, designed by Gomez et al., ended with a capture into a highly inclined two-body orbit about Europa. Russell recently located families of periodic orbits that exist around Europa in the Jupiter-Europa three-body system. Rather than inserting into a two-body orbit, the methods presented here will be used to locate a transfer trajectory between a Jupiter-Europa L2 halo orbit and a periodic orbit about Europa by connecting their respective invariant manifolds.

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**REFERENCES**


