LIAISON NAVIGATION IN THE SUN-EARTH-MOON FOUR-BODY PROBLEM

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Liaison Navigation involves the use of scalar satellite-to-satellite tracking data to autonomously determine both the relative and absolute positions and velocities of a constellation of spacecraft. It was shown that Liaison Navigation worked well for spacecraft in halo orbits in the Earth-Moon three-body problem. In this paper, Liaison Navigation is simulated using libration orbits in the bicircular four-body problem involving the Earth, the Moon, and also the Sun. These bicircular libration orbits similar to halo orbits were computed using a multiple shooting method and initialized with a halo orbit computed in the three-body problem. Constellations of two spacecraft were placed in various bicircular orbits and crosslink range measurements were simulated. With those observations, orbit determination was used, and the resulting covariance matrix gave an estimate of the orbit determination accuracy. The fourth body gravity introduced large variation in the orbit determination accuracy which depended on the positions of the primary bodies. The average increase in error was about 3% over similar three-body halo orbits. When the Earth-Moon orbit plane is inclined with respect to the Sun-Earth orbit plane, the error was increased again by about 1%. Solar Radiation Pressure was included in the force model with no significant change in orbit determination accuracy. Estimating the reflectance of both spacecraft in a two-spacecraft constellation was possible to within 1-2% of the true reflectance. It appears that orbit determination error on the order of 10 m (1σ) could be achievable in actual operation.

INTRODUCTION

Autonomous orbit determination is performed with measurements obtained using only the equipment on board orbiting spacecraft. For interplanetary missions, autonomy would alleviate some of the burden on the Deep Space Network (DSN). A new type of autonomous orbit determination was proposed called Linked, Autonomous, Interplanetary Satellite Orbit Navigation (LiAISON). LiAISON, hereafter referred to as Liaison Navigation, is a method where two or more spacecraft can determine their orbits using only Satellite-to-Satellite tracking (SST) data, which are observations taken along the line-of-sight vector between two spacecraft such as crosslink range and range-rate. SST provides information on the size, shape, and relative orientation of the orbits of the spacecraft being tracked. Suppose an orbit exists with a unique size and shape and it has only one possible absolute orientation in space. If a spacecraft were following this unique orbit and SST observations could be used to correctly determine the size and shape of the orbit, it follows that since there is only one possible orientation for an orbit

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of that size and shape, the absolute position of the spacecraft on the unique trajectory could be known and no ground-based tracking would be needed. Since SST also provides information on the relative orientation of the trajectories of the other spacecraft involved, the absolute positions of all the other spacecraft participating in the SST observations would be known as well. Thus, if one spacecraft is in a unique, or locally unique, trajectory, SST observations generated between it and another spacecraft could be used to estimate the absolute positions of both. That is the definition of Liaison Navigation: using only SST to determine the orbits of two or more participating spacecraft when one of them is in an orbit of a locally unique size and shape.

Libration orbits in the three-body problem are locally unique, which makes them ideal test beds for investigating how well Liaison Navigation works. Halo orbits generated using the circular restricted three-body problem (CRTBP) are among the simplest libration orbits to create. In a previous study, simulations of two spacecraft in lunar halo orbits showed that Liaison Navigation can be used to observe both sets of orbit state parameters as well as a range bias. That study also found these important principles affecting the accuracy of Liaison Navigation:

- Spacecraft should have relatively large separation distances.
- Constellation orbits should not be coplanar or nearly so.
- Constellations with shorter orbit periods lead to quicker convergence.
- Constellations with more spacecraft lead to quicker convergence.

This paper continues that study by investigating the effects of a fourth body on the orbit determination accuracy. In this case, the orbits are still lunar halo orbits, but the new fourth body is the Sun. Solar gravity and radiation pressure are added to the force model when generating the halos. The gravitational model used is called the bicircular model.

**BICIRCULAR MODEL**

In the bicircular model, the Earth and Moon travel in circles about their mutual barycenter. The Sun travels in another circle about the Earth-Moon barycenter. Usually, this is done in a rotating frame so that the Earth and Moon always lie on the x-axis of the frame, with the Sun still traveling in a circle about the origin with the proper period. A slight variation on the bicircular model was used for this study. It can be called the “inertial bicircular model.” In the inertial bicircular model, the Sun and the Earth-Moon barycenter travel in circles about the Sun-Earth/Moon barycenter. The Earth and Moon then travel in circles about the Earth-Moon barycenter. This was done to make the frame inertial, which simplifies the acceleration terms due to the gravity of each body. The inertial bicircular model also will make it more straightforward when adding in changes in the geometry of the primary body orbits. Figure 1 shows an illustration of the inertial bicircular model.

In Figure 1, the origin is at the Sun-Earth/Moon barycenter. B is the angle measured from the x-axis of the inertial frame to the line from the Sun to the Earth-Moon barycenter. M is the angle between the x-direction and the line joining the Earth and Moon. MB is the angle between the Earth-Moon line and the line between the Earth-Moon barycenter and the Sun. All angles are positive in the counter-clockwise direction.
The position of the Earth-Moon barycenter is computed with the following equation.

\[ \vec{r}_B = r_\odot (1 - \mu_3) \begin{bmatrix} \cos(B) & \sin(B) & 0 \end{bmatrix}^T \]

\( \vec{r}_B \) is the vector position of the Earth-Moon barycenter, \( r_\odot \) is the length of the semimajor axis of the Earth’s orbit about the sun, and \( \mu_3 \) is the gravitational parameter for the three-body system including the Sun and the Earth/Moon Barycenter. It is computed as follows:

\[ \mu_3 = \frac{\mu_\odot + \mu_\oplus}{\mu_\odot + \mu_\oplus + \mu_\odot} \]

The position of the Sun is computed with this equation:

\[ \vec{r}_\odot = -r_\odot \mu_3 \begin{bmatrix} \cos(B) & \sin(B) & 0 \end{bmatrix}^T \]

The positions of the Earth and Moon are found with these equations:

\[ \vec{r}_\oplus = \vec{r}_B + (1 - \mu_2) r_\oplus \begin{bmatrix} \cos(M) & \sin(M) \end{bmatrix}^T \]

\[ \vec{r}_\odot = \vec{r}_B - \mu_2 r_\oplus \begin{bmatrix} \cos(M) & \sin(M) \end{bmatrix}^T \]

\( \mu_2 \) is the gravitational parameter for the three-body system which includes the Earth and the Moon:

\[ \mu_2 = \frac{\mu_\oplus}{\mu_\odot + \mu_\oplus} \]

\( r_\oplus \) is the length of the semimajor axis of the Moon’s orbit about the Earth.

Since the Moon’s orbit about the Earth is not exactly in the ecliptic plane, we wanted to be able to adjust the inertial bicircular model accordingly to see what would happen to the orbit determination.

Figure 1 Geometry of the Inertial Bicircular Model.
Figure 2 Geometry of the inclined Inertial Bicircular Model.

accuracy. To do this, further adjustments were made to the bicircular model so that the Earth and Moon traveled in circular orbits that were inclined with respect to the plane of the ecliptic, which in this case is the plane of the Sun and the Earth/Moon barycenter. Figure 2 shows how this was done.

Figure 2 is similar to Figure 1, except $\Omega$ is the angle between the x-direction and the ascending node of the Moon. At this point, the Moon’s orbit ascends through the ecliptic with inclination $i$. The ecliptic in this case is the plane containing the Sun and the Earth-Moon barycenter. $M$ is the angle from the ascending node to the line joining the Earth and Moon.

The positions of the Earth-Moon barycenter and the Sun are computed using the equations listed previously. The positions of the Earth and Moon are computed, then rotated first about the x-direction by the angle $i$ and then about the z-direction by the angle $\Omega$. The result is as shown below.

$$\bar{r}_\odot = \bar{r}_B + (1 - \mu_2) \bar{r}_{\oplus} \begin{bmatrix} \cos(-\Omega) \cos(M) + \sin(-\Omega) \cos(-i) \sin(M) \\ -\sin(-\Omega) \cos(M) + \cos(-\Omega) \cos(-i) \sin(M) \\ \sin(-i) \sin(M) \end{bmatrix}$$

$$\bar{r}_{\oplus} = \bar{r}_B - \mu_2 \bar{r}_\odot \begin{bmatrix} \cos(-\Omega) \cos(M) + \sin(-\Omega) \cos(-i) \sin(M) \\ -\sin(-\Omega) \cos(M) + \cos(-\Omega) \cos(-i) \sin(M) \\ \sin(-i) \sin(M) \end{bmatrix}$$

In orbit determination tests, results from the three-body problem and the bicircular model were to be compared. To keep units and numbers consistent in ephemerides for both models, the bicircular
accelerations were converted to three-body accelerations by making the spacecraft’s acceleration due to the Sun’s gravity equal to the acceleration of the Earth-Moon barycenter. This effectively negates the effect of the Sun on the spacecraft.

Finally, some orbits were propagated with the effect of Solar Radiation Pressure (SRP) included as well. The acceleration due to SRP is

\[
\ddot{r}_{SRP} = p_{SR} c_R \frac{A_\odot}{m} \frac{\hat{r}_{\odot, sat}}{|\hat{r}_{\odot, sat}|^2}
\]

c\(_R\) is the reflectivity of the spacecraft. \(A_\odot\) is the cross-sectional area of the spacecraft facing the Sun in square meters. \(m\) is the mass of the spacecraft in kilograms. \(\hat{r}_{\odot, sat}\) is the vector from the center of the Sun to the spacecraft. \(p_{SR}\) is the pressure of solar radiation in Pa. This is about \(4.51 \times 10^{-6}\) Pa at the Earth. For varying distances from the Sun, \(p_{SR}\) would be

\[
p_{SR} = 4.51 \times 10^{-6} \text{ Pa} \left(\frac{(149,597,870 \text{ km})^2}{|\hat{r}_{\odot, sat}|^2}\right)
\]

\(\hat{r}_{\odot, sat}\) should be in km. A factor \(\ell\) can be applied to represent the fraction of the Sun’s face that is visible. When \(\text{km}\) and \(\text{km/s}\) are the units used for position and velocity, respectively, the acceleration due to the Sun’s gravity and SRP can be combined, resulting in

\[
\ddot{r}_{SRP+\odot} = \left[\ell c_R \frac{A_\odot}{m} (149,597,870 \text{ km})^2 (4.51 \times 10^{-6} \text{ Pa}) \left(\frac{1 \text{ km}}{1000 \text{ m}} - \mu_\odot\right) \frac{\hat{r}_{\odot, sat}}{|\hat{r}_{\odot, sat}|^3}\right]
\]

\(\ell = 1\) when there is a clear line of sight vector between the spacecraft and the Sun, and \(\ell = 0\) when the Sun’s light is blocked by a body. When determining the line of sight, it was assumed that the Moon would be the only body that would eclipse the Sun. The angle between the center of the Moon and the center of the Sun was called \(\varsigma\) and is found using the dot product.

\[
\varsigma = \arccos \left(\frac{\hat{r}_{\odot, sat} \cdot \hat{r}_{\odot, sat}}{|\hat{r}_{\odot, sat}|^2 |\hat{r}_{\odot, sat}|}\right)
\]

\(\hat{r}_{\odot, sat}\) is the vector from the center of the Sun to the spacecraft and \(\hat{r}_{\odot, sat}\) is from the Moon to the spacecraft. \(|\hat{r}_{\odot, sat}|\) and \(|\hat{r}_{\odot, sat}|\) are the magnitudes of those vectors. The angle between the center and limb for the two bodies are found using these equations, assuming that the bodies are spherical.

\[
\varsigma_\odot = \arctan \left(\frac{R_\odot}{|\hat{r}_{\odot, sat}|}\right)
\]

\[
\varsigma_\odot = \arctan \left(\frac{R_\odot}{|\hat{r}_{\odot, sat}|}\right)
\]

\(R_\odot\) and \(R_\odot\) are the radii of the primary bodies. The relations below were used in determining \(\ell\).

\[
\ell = \begin{cases} 
1, & \varsigma \geq \varsigma_\odot + \varsigma_\odot \\
\frac{\varsigma - \varsigma_\odot}{2\varsigma_\odot} + \frac{1}{2}, & \varsigma_\odot - \varsigma_\odot \leq \varsigma < \varsigma_\odot + \varsigma_\odot \\
0, & \varsigma < \varsigma_\odot - \varsigma_\odot 
\end{cases}
\]

This is based on the assumption that \(\varsigma_\odot \gg \varsigma_\odot\). If that is false, it only negatively affects the computation of \(\ell\) when the Sun is partially eclipsed.
MULTIPLE SHOOTING METHOD

In the four-body problem, there are no longer any periodic solutions because the same positions of the primaries do not repeat within any reasonable length of time. This means that the periodic halo orbits found in the three-body problem do not occur in the four-body problem. Instead, trajectories must be computed that are fairly close to periodic, at least for the time interval desired. These quasi-periodic orbits are a subset of Lissajous orbits. A numerical method used to find a continuous libration trajectory in the four-body problem is a multiple shooting method adapted from the work of Howell, Pernicka, and Wilson.⁵,⁶

To initialize the Multiple Shooting Method, a reference trajectory is required. For these “initial guess” trajectories, we used a halo orbit from the CRTBP which had been rotated and scaled to fit the inertial bicircular frame. Patch points were selected along the halo so that there were four patch points per halo orbit period with a certain time interval between them. A trajectory segment is propagated using the inertial bicircular model starting at each patch point for the proper amount of time. These trajectory segments won’t match up at the end points due to the added effect of the Sun’s gravity. Using two levels of iteration, the initial conditions of each segment are corrected until they can be patched together into a continuous trajectory.

Level 1

In the first level of iteration, the velocity at each patch point is corrected until the end positions on each trajectory segment match the start positions of the succeeding segment. Putting all the segments together result in a trajectory that is continuous in position, but not in velocity. There will be velocity discontinuities or Δv’s at each interior patch point.

Level 2

The Δv’s at each interior patch point are reduced in the second level of iteration by adjusting the positions and times of the patch points. After correcting the positions and times of the patch points, the Level 1 iteration must be repeated to make the trajectory continuous in position. After the Δv’s reach a sufficiently small value, the trajectory is considered continuous.

Implementation of Multiple Shooting

When using the Multiple Shooting Method, the positions and times at the exterior patch points of the trajectory are usually adjusted to a much greater extent than the interior patch points. This results in some “end effects” in the trajectories where the spacecraft would quickly fall away from the libration point if the trajectory were propagated away from the end points. For this study, libration orbits of one halo orbit period were desired. To eliminate end effects, the reference trajectory used in the Multiple Shooter consisted of three complete revolutions of the halo orbit. After converging on a continuous trajectory, the first and last revolutions of the Lissajous orbit were discarded and the center portion of the orbit was used to create an ephemeris file.
Ephemerides for Orbit Determination

The halo orbits used as initial conditions for finding quasi-periodic halo orbits in the bicircular model were computed using the technique explained by Howell.\(^7\) Families of halo orbits were generated around the Lunar \(L_1\) and \(L_2\) Lagrange points (\(LL_1\) and \(LL_2\)) in the CRTBP. For each family of halo orbits around a Lagrange point, a certain number were selected as test orbits and ephemeris files were generated. Figures 3 and 4 show the distribution of the initial conditions of the test halo orbits. The halo orbits shown that have a “Northern” orientation were rotated and scaled into the inertial bicircular frame to initialize the Multiple Shooting Method.

An example of a quasi-periodic orbit obtained from the Multiple Shooting Method is shown next to the CRTBP halo used to generate it in Figure 5. Both are displayed in the rotating frame of the CRTBP, where a length unit is equal to the distance between the Earth and the Moon. \(\tau\) will be described later in the paper. The orbits are plotted for slightly more than one halo orbit period and they are very similar. It might be possible to see that the bicircular orbit does not exactly repeat. The orbit does not come back exactly to its initial position and is slightly offset in the upper middle portion of the left side.

![Figure 3 Initial conditions and names of LL$_1$ test orbits.](image)
Figure 4 Initial conditions and names of LL₂ test orbits.

Figure 5 Halo orbits computed in the CRTBP and the Bicircular model using LL1B with \( \tau = 0 \) and \( MB(t_0) = 0 \) and projected into the x-y plane.
ORBIT DETERMINATION METHODS

Orbit determination attempts to estimate the initial conditions of the nonlinear system of differential equations describing spacecraft motion using measurements that are nonlinear with respect to the state. This is done by linearizing about a reference orbit that is close to the actual orbit. Instead of estimating the initial state with the measurements, the error in the initial state is estimated using the observation residuals, which are the observed measurements minus those computed from the reference solution. Liaison Navigation can be conducted using any type of orbit determination method. However, in this study the technique used was the batch processor. A summary of this technique comes from Tapley, et al., and uses their notation. The equations are shown for two spacecraft.

The spacecraft state consists of three position and velocity components for each spacecraft. For two spacecraft, it would be

\[
\mathbf{X} = \begin{bmatrix} x_1 & y_1 & z_1 & \dot{x}_1 & \dot{y}_1 & \dot{z}_1 & x_2 & y_2 & z_2 & \dot{x}_2 & \dot{y}_2 & \dot{z}_2 \end{bmatrix}^T,
\]

where the subscripts denote the spacecraft number. The equations of motion are written

\[
\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}, t), \quad \mathbf{X}_i = \mathbf{X}(t_i).
\]

An observation at time \( t_i \) is denoted by \( \mathbf{Y}_i \), and the state deviation and observation residuals are written

\[
\mathbf{x}(t) = \mathbf{X}(t) - \mathbf{X}^*(t), \quad \mathbf{y}(t) = \mathbf{Y}(t) - \mathbf{Y}^*(t),
\]

where \( \mathbf{X}^* \) is the reference solution, or the “best guess” orbit, and \( \mathbf{Y}^* \) is an observation computed using the reference solution. The observations can be related to the state with the \( \mathbf{H} \) matrix, which is

\[
\mathbf{H}_i = \left[ \frac{\partial \mathbf{G}}{\partial \mathbf{X}} \right]^*_i.
\]

\( G \) is a function that computes observations from a state vector in the following way:

\[
\mathbf{Y}_i = G(\mathbf{X}_i, t_i) + \epsilon_i.
\]

\( \epsilon_i \) is the observation error. If crosslink range, \( \rho \) is the observations types then \( \mathbf{H}_i \) is of the form:

\[
\mathbf{H}_i = \begin{bmatrix}
\frac{\partial \rho}{\partial x_1} & \frac{\partial \rho}{\partial y_1} & \frac{\partial \rho}{\partial z_1} & \frac{\partial \rho}{\partial \dot{x}_1} & \frac{\partial \rho}{\partial \dot{y}_1} & \frac{\partial \rho}{\partial \dot{z}_1} & \frac{\partial \rho}{\partial x_2} & \frac{\partial \rho}{\partial y_2} & \frac{\partial \rho}{\partial z_2} & \frac{\partial \rho}{\partial \dot{x}_2} & \frac{\partial \rho}{\partial \dot{y}_2} & \frac{\partial \rho}{\partial \dot{z}_2}
\end{bmatrix}_i
\]

State deviations at time \( t \) may be mapped back to the initial epoch, \( t_k \), with the state transition matrix \( \Phi \) in the following way:

\[
\mathbf{x}(t) = \Phi(t, t_k)\mathbf{x}_k.
\]

The state transition matrix is integrated along with the state using the following relation:

\[
\dot{\Phi}(t, t_k) = \mathbf{A}(t)\Phi(t, t_k) \quad \text{where} \quad \mathbf{A}(t) = \frac{\partial \mathbf{F}(\mathbf{X}^*, t)}{\partial \mathbf{X}(t)} \quad \text{and is of the form:}
\]
Relating an observation at time $t_i$ back to the state initial epoch $t_k$ is done with the $H_i$ matrix:

$$H_i = \tilde{H}_i\Phi(t_i, t_k)$$

So $H_i$ is the observation-state relationship at $t_i$ mapped back to the initial epoch $t_k$. If there are $\ell$ observations, all of them can be combined into a vector, and all of the $H_i$ matrices into a larger matrix so that

$$y = Hx(t_k) + \epsilon \quad \text{where} \quad y \equiv \begin{bmatrix} y_1 \\ \vdots \\ y_\ell \end{bmatrix} \quad \text{and} \quad H \equiv \begin{bmatrix} H_1 \\ \vdots \\ H_\ell \end{bmatrix} \quad \text{and} \quad \epsilon \equiv \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_\ell \end{bmatrix}$$

A least squares solution for $x(t_k)$ is computed with the equation

$$\hat{x} = (H^TH)^{-1}H^Ty.$$ 

If it is assumed that the observation errors can be modeled as white noise, and the standard deviation in the range noise is denoted $\sigma_\rho$, a weighting parameter $W$ can be used to weight the observations such that

$$\hat{x} = (H^TWH)^{-1}H^TWy,$$

where

$$W = \frac{1}{\sigma_\rho^2}$$

The matrices $H^TWH$ and $H^TWy$ can be accumulated one observation at a time:

$$H^TWH = \sum_{i=1}^{\ell} H_i^TWH_i \quad \text{and} \quad H^TWy = \sum_{i=1}^{\ell} H_i^TWy_i.$$

$H^TWH$ is also called the information matrix, $\Lambda$. The variance-covariance matrix is the inverse of the information matrix:

$$P = \Lambda^{-1}.$$ 

The covariance matrix can be used to compute the variances of the state parameters as well as the correlations between state parameters. In this study, the method used to compare the accuracy of orbit estimates was to propagate the batch covariance matrix over the entire data arc with the relation

$$P_i = \Phi(t_i, t_k)P_k\Phi^T(t_i, t_k).$$
Then the length of the largest axis of the 3D, 3σ position error ellipsoid for the spacecraft was computed. This was the maximum eigenvalue of the 3 × 3 position portion of the covariance matrix for that spacecraft. It will be called \( \beta_i \) at time \( t_i \).

\[
\beta_i = \max \left( \sqrt{\lambda_j} \right) \quad \text{where} \quad \lambda_j \quad \text{for} \quad j = 1, \ldots, 3 \quad \text{are the eigenvalues of} \quad [P_i]_{3 \times 3}.
\]

The average value of \( \beta_i \) over the entire data span will be called \( \bar{\beta} \):

\[
\bar{\beta} = \frac{\sum_{i=1}^{n} \beta_i}{n} \quad \text{where there are} \quad n \quad \text{values of} \quad \beta_i \quad \text{in the data arc.}
\]

The values of \( \bar{\beta} \) for both satellites can be averaged to produce a metric called \( \beta_{\text{ave}} \) that gives an indication of the overall navigation accuracy for the constellation for those particular conditions.

All of the constellations consisted of two spacecraft in bicircular orbits which were generated from the same three-body halo orbit. The two spacecraft were separated by a certain phase angle, \( \tau \). \( \tau \) is defined in the CRTBP and is a non-geometric angle describing the positions of the spacecraft along the halo orbits in a manner similar to mean anomaly. In the CRTBP, the coordinate frame rotates so that the planetary bodies are always on the x-axis, with the positive x-direction going from the larger to the smaller. The positive y-axis is parallel to the velocity vector of the smaller body. The phase angle, \( \tau \), is zero when the spacecraft is on the halo orbit where it intersects the x-z plane traveling in the positive y-direction. The phase angle increases in the direction of the orbital motion and is defined as

\[
\tau_i = \frac{(t_i - t_0)}{p}2\pi,
\]

where \( (t_i - t_0) \) is the time past the x-z plane crossing (at \( \tau = 0 \)) and \( p \) is the halo orbit period. Figure 6 shows some values of \( \tau \) on a halo orbit which has been projected onto the x-y plane.

With both spacecraft starting at different phase angles in these two-spacecraft constellations, the difference in phase angle, \( \Delta \tau = \tau_2 - \tau_1 \), remains constant.

The value of the error metric, \( \beta_{\text{ave}} \), varies with differences in \( \tau \), so it became necessary to find one representative value of \( \beta_{\text{ave}} \) for each combination of orbits. This metric will be called \( \beta_{\text{con}} \) and is used to rank the navigation accuracy of the different constellations.
TEST METHODS

To determine how the inclusion of a fourth body affected the orbit determination accuracy, a covariance analysis was conducted using simulations of two spacecraft. These two spacecraft were always placed on the same halo orbit, with different phase angles. Since the Northern and Southern halo orbits are symmetric, only the Northern halo orbits were used. The two spacecraft tracked each other continuously, unless the Earth or the Moon blocked the line of sight (LOS) vector. Range observations were generated from the ephemeris files every six minutes. These observations were used to generate the $H_i$ matrices. The state transition matrices were obtained from the ephemeris files for each orbit. The information matrix, $H^T W H$, was computed and the covariance matrix was used to compute navigation error metrics without actually estimating the state. The range observations had an added white noise component with a standard deviation of 1 m. The fit span for these tests was one halo orbit period.

Bicircular

For the bicircular constellations, the values of $\tau$ and the positions of the primaries were computed using the following equations:

\[
\begin{align*}
\tau_1 &= 0 \\
\tau_2 &= \begin{bmatrix} 30 & 60 & 90 & 120 & 150 & 180 & 210 & 240 & 270 & 300 & 330 \end{bmatrix} \\
S(t_0) &= 0 \\
MB(t_0) &= \begin{bmatrix} 0 & -45 & -90 & -135 & -180 & -225 & -270 & -315 \end{bmatrix}
\end{align*}
\]

All angles are shown in degrees. $S(t_0)$ is the value of angle $S$ at time $t_0$, where $t_0$ is the initial epoch in the ephemeris. Similarly, $MB(t_0)$ is the value of angle $MB$ at time $t_0$. Since there are eleven
different values of $\tau_2$ and eight of $MB(t_0)$, for each two-spacecraft constellation there were 88 different combinations for orbit determination runs. Choosing $\beta_{con}$ from those 88 different runs was performed by averaging the values of $\beta_{ave}$ for each $\tau_2$. The lowest average was chosen to be $\beta_{con}$.

For the CRTBP constellations, the halo orbits were propagated by nullifying the effect of Sun on the spacecraft as described previously. Since the positions of the primaries don’t matter, the values of $\tau$ were computed using the following equations:

$$\tau_1 = 0$$
$$\tau_2 = \begin{bmatrix} 30 & 60 & 90 & 120 & 150 & 180 & 210 & 240 & 270 & 300 & 330 \end{bmatrix}$$

Angles are shown in units of degrees. For the CRTBP runs, there were only 11 runs computed. $\beta_{con}$ for the CRTBP runs was the lowest value of $\beta_{ave}$.

**Inclined Bicircular**

For the inclined bicircular constellations, the CRTBP halo orbits LL1B and LL1C were used to generate the ephemerides for use in orbit determination. The values of $\tau$ and the positions of the primaries were computed using the following equations:

$$\tau_1 = 0$$
$$\tau_2 = 120$$
$$S(t_0) = 0$$
$$M(t_0) = \begin{bmatrix} 0 & 30 & 60 & 90 & 120 & 150 \end{bmatrix}$$
$$\Omega = \begin{bmatrix} 0 & 30 & 60 & 90 & 120 & 150 \end{bmatrix}$$

$M(t_0)$ is the value of angle $M$ in Figure 2 at time $t_0$. $\Omega$ is the value of that angle in the same figure. It was found that the orbit determination results when $M(t_0)$ and $\Omega$ were between 180 and 360 degrees, were almost identical to when they were between 0 and 180 degrees. Because of that, the results were not computed between 180 and 360 degrees. Since there are six different values of $M(t_0)$ and six of $\Omega$, for each two-spacecraft constellation there were 36 different orbit determination runs. Choosing $\beta_{con}$ from those 36 different runs was performed by averaging all the values of $\beta_{ave}$.

This process was performed for inclinations of 0, 1, 2, 5, and 10 degrees. The effect of the inclination could be seen by comparing the results for inclined orbits to the zero-inclination case.

**Solar Radiation Pressure**

To test the effect of SRP on the orbit determination accuracy, a pair of ephemerides was created using equations of motion that included SRP accelerations in the Multiple Shooting Method. These orbits were initialized using the LL1B halo orbit at $\tau_1 = 0^\circ$ and $\tau_2 = 120^\circ$ and the inclined inertial bicircular model with a $5^\circ$ inclination between the Earth-Moon orbit and the Sun-Earth orbit. The initial positions of the primaries were found using $M(t_0) = 0$ and $\Omega = 0$. The fit span was 1 halo orbit period, with observations every six minutes. The standard deviation of the noise on the range observations was 1 m. $c_R$ was 1.5, $A$ was 5 m$^2$, and $m$ was 1000 kg.
RESULTS

Bicircular

When comparing the values of $\alpha_{ave}$ for the CRTBP and the inertial bicircular model, there was a great degree of variability. Depending on the geometry of the primary bodies, $\alpha_{ave}$ for the CRTBP could be larger or smaller than for the inertial bicircular model. Table 1 shows the average percent increase or decrease in $\alpha_{ave}$. It also shows the average standard deviation of $\alpha_{ave}$ in the bicircular model. These values of standard deviation were computed separately for each $\Delta\tau$ and then averaged. The standard deviations are displayed as percentages of $\alpha_{con}$ for the CRTBP. As an example of how to read the table, a constellation of two spacecraft in the bicircular LL1D halo orbit would have orbit determination results about 0.41% better than the CRTBP on average, but the accuracy would vary $\pm$12% of 39.9 m as the geometry of the Sun, Earth, and Moon changed. Averaging it all together, the bicircular $\beta$’s are 3.3% larger than for CRTBP with a standard deviation of about 15%. As stated before, these results are very rough due to the large variability.

<table>
<thead>
<tr>
<th>Orbit Name</th>
<th>CRTBP $\alpha_{con}$ (m)</th>
<th>Bicircular change in $\alpha_{ave}$ (%)</th>
<th>Bicircular std. dev. of $\alpha_{ave}$ (% of CRTBP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LL1A</td>
<td>29700</td>
<td>2.6</td>
<td>13</td>
</tr>
<tr>
<td>LL1B</td>
<td>136</td>
<td>2.4</td>
<td>9.0</td>
</tr>
<tr>
<td>LL1C</td>
<td>139</td>
<td>6.4</td>
<td>15</td>
</tr>
<tr>
<td>LL1D</td>
<td>39.9</td>
<td>-0.41</td>
<td>12</td>
</tr>
<tr>
<td>LL1E</td>
<td>106</td>
<td>-16</td>
<td>21</td>
</tr>
<tr>
<td>LL1F</td>
<td>158</td>
<td>16</td>
<td>27</td>
</tr>
<tr>
<td>LL2A</td>
<td>26500</td>
<td>-0.76</td>
<td>8.5</td>
</tr>
<tr>
<td>LL2B</td>
<td>70.6</td>
<td>-0.59</td>
<td>7.6</td>
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<tr>
<td>LL2C</td>
<td>63.1</td>
<td>0.84</td>
<td>9.0</td>
</tr>
<tr>
<td>LL2D</td>
<td>56.6</td>
<td>14</td>
<td>37</td>
</tr>
<tr>
<td>LL2E</td>
<td>92.6</td>
<td>12</td>
<td>66</td>
</tr>
<tr>
<td>LL2F</td>
<td>353</td>
<td>2.7</td>
<td>18</td>
</tr>
</tbody>
</table>

Inclined Bicircular

The LL1B and LL1C orbits were selected for tests involving inclination between the plane with the Sun and Earth/Moon barycenter and the Earth-Moon orbit plane. Since the Moon’s orbit is inclined about 5° with respect to the ecliptic, a range of inclinations were chosen around this value. It was found that the higher the inclination, the worse $\beta$ turned out to be. Table 2 shows that $\beta_{ave}$ increased by about 1% for the 5° inclination as compared to the 0° inclination. The standard deviation of $\beta_{ave}$ stayed at similar values for all inclinations. The percent change at 0° inclination in Table 2 doesn’t match the percent change in Table 1 because the ephemerides for this comparison were only generated using a $\Delta\tau$ of 120° and different Sun-Earth-Moon geometries than before.
Table 2 COMPARISON OF ORBIT DETERMINATION ACCURACY FOR DIFFERENT INCLINATIONS OF THE MOON’S ORBIT PLANE.

<table>
<thead>
<tr>
<th>Inclination change in LL1B $\beta_{con}$ (% of CRTBP)</th>
<th>change in LL1C $\beta_{con}$ (% of CRTBP)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.36</td>
</tr>
<tr>
<td>1</td>
<td>2.09</td>
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<tr>
<td>2</td>
<td>1.82</td>
</tr>
<tr>
<td>5</td>
<td>0.992</td>
</tr>
<tr>
<td>10</td>
<td>0.486</td>
</tr>
</tbody>
</table>

Solar Radiation Pressure

With an infinite a priori variance on $c_R$, the resulting standard deviations on the $c_R$ estimates were 0.023 for the spacecraft at $\tau_1 = 0$, and 0.029 for the spacecraft at $\tau_1 = 120$. This seems to be fairly good, since those represent only 1-2% of the total $c_R$. Even though $c_R$ estimates were fairly tight, $\beta_{ave}$ went from 112.96 m without SRP to 243.79 m with SRP. When $c_R$ is not estimated, the effect of SRP on the orbit determination accuracy is negligible. In that case, $\beta_{ave}$ was 112.93 m.

Final Simulation

A final run was performed that was meant to simulate as close as possible the orbit accuracy attainable in a real operations environment. The first orbit was the LL1B halo at $\tau_1 = 120$ in the inclined inertial bicircular model. $M(t_0)$ and $\Omega$ were zero degrees. The second orbit was a lunar orbit propagated using the same model. The initial conditions were obtained using Keplerian elements, which were converted to Cartesian position and velocity. The Keplerian orbit was circular with an altitude of 5,000 km above the Moon. The inclination was 90° and the longitude of the node, argument of perigee, and true anomaly were all zero. Both orbits were computed using SRP. $c_R$ was 1.5 and $A/m$ was 5 m²/1000 kg for both spacecraft. The interval between observations was six minutes. Consider covariance analysis was performed for a sinusoidal observation error. The modeled amplitude of this error was zero, but a 1σ uncertainty of 50 cm was considered. The range bias and reflectances were estimated along with the positions and velocities of the spacecraft. Observations were generated in two-hour periods which were separated by two-hour periods with no observations. The standard deviation on the range measurement noise was set to 1 m. The fit span was half of a LL1B halo orbit period, or about six days. During that time, the maximum range between the spacecraft was about 64,000 km, and the average range was about 55,000 km. $\beta_{ave}$ was 24.1 m. The $\beta$ for the satellite in the halo was 38.5 m and 9.9 m for the satellite in lunar orbit. Remember that these $\beta$ values are 3σ. The number of observations during the fit span was 711, with 6 observations blocked by the Moon. The standard deviation for the range bias estimate was 2.7 m. The standard deviations for the reflectance parameters for spacecraft one and two were 0.02 and 0.01 respectively.

The results of this final run showed that Liaison Navigation still performs very well when a fourth body and SRP are added to the force model. In addition, the range bias and reflectance parameters were estimated to good precision. It seems like an average 1σ orbit determination accuracy on the order of 10 m might be possible for Liaison Navigation in actual mission operations.
CONCLUSION

When trying to decipher the effect of a fourth body on Liaison Navigation accuracy, the only certain conclusion is that the fourth body introduces large variability in accuracy as the primary bodies shift positions. On average, the effect of the Sun’s gravity on Earth-Moon halo orbits seems to increase the orbit determination error on the order of 3%, while introducing fluctuations with standard deviations between approximately 10 and 40% of the mean. The effect of the inclination between the Sun-Earth orbit plane and the Earth-Moon orbit plane also added about 1% to the orbit determination error. SRP itself does not seem to effect the dynamics enough to cause an increase in the orbit determination error, although the estimation of the reflectance ($c_R$) caused the error metric to increase.

These small differences between the CRTBP and the bicircular model show that the CRTBP should be a reasonable approximation when estimating orbit determination accuracy. It appears that Liaison Navigation could be a very useful tool for orbit determination in libration point orbits.

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REFERENCES


