RECURSIVE NONLINEAR ESTIMATION FOR
RELATIVE NAVIGATION IN ELLIPTICAL ORBITS

by

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The final copy of this thesis has been examined by the signators, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.
Several nonlinear recursive suboptimal estimators are applied to the problem of determining the relative motion of one satellite with respect to another, both in close elliptical orbits. Only relative measurements of range, range rate and angles are used. The two-step filter, iterated extended Kalman filter, and a Kalman filter using a change of variables that makes the measurement equation linear are all derived from different approximations to the same global least squares cost function. The potential for the two-step estimator to generate numerically rank deficient covariance matrices is shown. A test matrix composed of the partial derivatives between the first and second step states and a set of constant vectors is derived to identify points in which this anomaly could occur. This rank deficiency can be interpreted geometrically in the first step state space. It is shown that the existence of low eigenvalues of this covariance matrix does correspond to situations in which the first step state vector has very low error in some direction in that state space. A simple example problem is used to demonstrate these findings. It is also shown that a two-step filter defined with an equal number of first and second step states is the same as a Kalman filter using a change of variables, except that a better approximation is made to the first step covariance time update. The dynamics and measurement equations are derived for the elliptical orbit relative navigation problem. Additionally, a closed form state transition matrix for the linearized equations of motion is derived and used to assess the magnitude of the linearization errors. The two-step filters applied to this problem produced a lower mean square error than the other filters under two situations. The first is when a large observation vector was used giving good observability of the state, but
initialized with a large error. The second is for scalar observations and a statistically distributed set of initial errors.
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CHAPTER 1

INTRODUCTION

1.1 Background and Motivation

Several space missions have been proposed, are in development, or are operational which require coordination of two or more satellites in a highly elliptical orbit \((e > 0.7)\). One such class of mission involves a "cluster" of four satellites in a tetrahedron formation for the purpose of obtaining space physics measurements in different sections of the magnetosphere. The ESA Cluster [ESA, 1983], [Stephenson, 1991], [De Selding, 1997], the Russian Interball [Covault, 1995], and the proposed Grand Tour Cluster [Curtis, 1996] are examples of this class of missions. Rendezvous and docking in a geostationary transfer orbit, for the purposes of satellite assembly and repair, has been considered in some advanced systems studies [ESA, 1983], [Katzberg and Garrison, 1995]. This scenario involves similar dynamics and has similar mission requirements to the space physics missions.

For future missions like these, it would be advantageous to use an autonomous relative navigation system on one of the vehicles to estimate the differences in orbital states between it and the other satellite. This navigation filter could operate without continuous orbit determination information from ground sites in order to reduce operations costs and make the constellation maintenance more robust. Navigation sensors for this purpose include either intersatellite radar on one of the vehicles or Global Positioning System (GPS) receivers on both vehicles.

A radar system can provide observations of range, range rate, and in some
applications, angles. When using GPS in a highly elliptical orbit, there are large time periods in the vicinity of apogee where fewer than four satellites are visible and individual pseudoranges are the only observation available [Potti, et al., 1995], [Eissfeller, et al., 1996], [Garrison and Azelrad, 1996].

For both of these sensors, the estimation problem is characterized by frequent samples of a small-dimensioned observation vector. Observability of the complete orbital state is only possible by processing a sequence of measurements over time. The navigation filter and its dynamic model are therefore relied upon heavily to optimally combine the sequence of measurements and propagate the state estimate.

Relative motion in elliptical orbits is a nonlinear dynamic problem. No counterpart of comparable accuracy to the Hill's equations [Clohessey and Wiltshire, 1960] exists for orbits which have the large eccentricities (e > 0.6) considered in this study.

The measurements of range and range rate between the two satellites are also nonlinear functions of the orbit state. The GPS pseudorange equations are nonlinear functions of the orbit position as well, but are linear functions of the receiver clock bias. These two characteristics of the measurement equation: frequent update using a small dimensioned vector and nonlinear observation equations, indicate that advances in nonlinear estimation could offer improved techniques for relative navigation in elliptical orbits.

The objective of this research is a comparison between the existing nonlinear filtering techniques and a recent development in least squares estimation, the two-step filter. The relative navigation problem is the application which is used to make this comparison although it is expected that most of the findings are fundamental to the structure of the two-step filter and thus have a broader application.
1.2 Review of Previous Work

Approximations to the dynamics of relative motion in elliptical orbits have attempted to follow the classical work for the similar problem in circular orbits [Clohessey and Wiltshire, 1960]. Several notable articles dealing with motion relative to an arbitrary eccentricity reference orbit are [Bender and Blackford, 1968], [Tschauner and Hempel, 1965], and more recently [Carter, 1993]. Furthermore, recent research reported in [Zare, 1990], [Hujsak, 1989], [Ashenberg, 1994], [Alfano and Negron, 1993], and [Alfano, 1994] on the evolution of orbital debris fields has dealt with motion of objects in highly elliptical orbits. In particular, Ashenberg [Ashenberg, 1994] provides general expressions relating relative orbit elements to relative positions and velocities. His analysis seeks to facilitate the propagation of the error covariance associated with the debris field resulting from a satellite explosion in orbit. A series expansion in eccentricity to second order was done recently by [Melton, 1997]. His results indicated an accuracy of 10 to 20 percent for moderate eccentricities (0 < e < 0.2) and concluded that this accuracy was sufficient for mission planning and preliminary design purposes. None of these linear models are of sufficient fidelity to be used for long period dynamic models in a navigation filter orbits with large eccentricities considered in this study.

Having recognized the need to include the nonlinear dynamic equations and process nonlinear measurements, the classical results of linear filtering theory [Kalman, 1960], [Kalman, 1961] are no longer applicable. Conventional extensions of this theory to nonlinear problems include the extended Kalman filter, iterated extended Kalman filter, as well as series expansions which extend the approximations to the measurement and dynamic equations beyond first order [Maybeck, Vol. 2 1979]. Still another approach is the use of a change of variables which makes the measurement equation linear such that no approximations are
made in the state updates [Mehra, 1971], [Balakrishnan and Speyer, 1985],
[Balakrishnan, 1989]. All of these approaches, however, are applications of the
linear theory as an approximation of the true nonlinear problem. In many
applications this approximation is valid and the aforementioned estimators have
been utilized with success. The estimates are not guaranteed to be optimal and
often have a significant bias. The true optimal nonlinear estimation problem has
been studied theoretically by others [Kushner, 1967(a)], [Kushner 1967(b)],
[Daum, 1980] but has not been reduced to a practical filter. A practical recursive
filter has recently been proposed [Haupt, et al, 1996], [Kasdin, et al, 1997] which
provides a better, although still sub optimal, solution to the nonlinear dynamic
estimation problem. This “two-step” filter is in fact an optimal estimator for a
particular class of static problems. The dynamic estimator is derived from an
approximation to the static results.

1.3 Contributions of This Research

First, a new formulation of the linear approximation to the relative motion
problem in elliptical orbits is derived. This is used in early conceptual studies of
this problem to assess the magnitude of errors which result from linearization and
in the application of conventional linearized estimation techniques to this problem.
It is found that the linearized equations of motion are not an adequate model for
relative motion in elliptical orbits.

An analytical study is conducted on the differences between the two-step
estimator and other conventional suboptimal nonlinear filters all of which are
derived from various approximate minimizations of the same least-squares cost
function. This expands upon the existing knowledge by investigating a special case
of the two-step filter which has an equal number of first and second step states.
The fundamental improvement of the two-step filter over a
coordinate-transformation based Kalman filter is shown to lie in the more accurate approximation to the covariance matrix time update in the former. This finding is also verified through numerical simulations.

It is shown that the two-step estimator occasionally generates a covariance matrix with a very low or even a small negative eigenvalue. This can lead to numerical problems which cause filter divergence or sometimes complete failure to generate any physically meaningful solution. This problem is addressed analytically by making several approximations to the filter. The covariance matrix rank is then shown to be expressed in terms of the linear independence of a set of constant vectors and the partial derivative matrix computed along the state trajectory. These findings result in a test for the location of state trajectory points where these low eigenvalues can occur. In addition, this test matrix has a geometric interpretation in the abstract state space. This geometry is used to visualize the mathematical reason why the covariance matrix drops rank.

Direct modifications are then made to the existing two-step filter to prevent this condition from occurring. These findings are first applied to a number of example problems involving different measurement sets and state dimensions to illustrate the aforementioned properties and are then applied to the original relative navigation problem.

The truth model for the relative navigation problem is constructed. Several different filters are derived, using different state definitions and measurement sets. These demonstrate by numerical simulations the findings which were previously shown analytically. The first measurement set is one of good state observability. For this example the filter is initialized with a large state error. The second measurement set consists of one scalar measurement available at each time step and thus is dependent upon the filter to generate state observability. This case is simulated starting from an ensemble of randomly distributed initial conditions.
The numerical results are shown to validate the theoretical predictions.

1.4 Outline

Chapter 2 reviews nonlinear estimation theory including derivation of the two-step estimator. Chapter 3 presents the analysis into the existence and cause of rank deficient covariance matrices in the two-step estimator. Chapter 4 and 5 describe the truth model and estimator model, respectively. Chapter 4 also contains a linearized model of relative motion in elliptical orbits. Chapter 6 presents the results from numerical simulations of the relative navigation problem.

1.5 Definitions and Nomenclature

The two satellites used in the relative navigation problem will be referred to as the “primary” and the “secondary”. For a rendezvous and docking problem, the primary is the pursuer vehicle and the secondary is the target. The radar sensor is located on the primary and all relative motion is measured with respect to the primary’s center of mass. The notation \( - \) is used to indicate an \textit{a priori} estimate or condition, immediately prior to the update of the state estimate with the latest observation vector. The notation \( + \) is used to identify the \textit{a posteriori} conditions, immediately following the state update. The “hat” \( \hat{\cdot} \) is used to identify a variable as the estimate of a vector. Subscripts are used to identify individual components of a vector or a matrix if the same letter is used without the vector \( \hat{\cdot} \) or hat symbols. Time steps are indicated by subscripts of a vector and iteration counts are indicated by superscripts. For example, \( \hat{x}_i^k \) is the \( k^{TH} \) iteration of the estimate of \( \hat{x} \) at time \( t_i \). Hence, the estimate of \( \hat{x} \) at the time \( t_i \) following an update with the new observation at this time step is \( \hat{x}_i^k(+) \). The predicted value of the state \( \hat{x}_i \) at time \( t_i \) before the new observation is processed is \( \hat{x}_i(-) \). The third component of the vector \( \hat{x} \) is \( x_3 \). A lower case letter without a hat or vector symbol over it is a scalar. When discussing a specific coordinate system, a sequence of letters (such as
\{x, y, z, u, v, w\} will be used to identify the position and velocity in that system for the derivation of equations of motion to be used in the dynamic models and the simulation of the truth model. The aforementioned convention identifying elements of the state vector by subscripts (\{x_1, x_2, \ldots\}) will be used when the specific application to the filtering problem is discussed. The expected value of a random variable is indicated by \(\langle\rangle\). Unless stated otherwise, Matrices are identified by capital letters and use the same subscript and superscript convention as do vectors.

For the special case of a static estimation problem, the subscript 0 will be used in addition to the subscript indicating the time step. For these problems the estimate does not change between discrete measurements and therefore
\[ \hat{x}_{0,t+1}(-) = \hat{x}_{0,t}(+) \]. The notation \(\langle\rangle\) and \((-)\) is thus redundant and will be deleted. The estimate of the (constant) state vector \(\hat{x}_0\) following processing of the \(i^{TH}\) measurement is \(\hat{x}_{0,i}\). This is identical to \(\hat{x}_{0,i}(+)\) and \(\hat{x}_{0,i+1}(-)\) above.
CHAPTER 2

TWO-STEP FILTER

In this chapter the two-step estimator is derived. The differences between this estimator and other conventional nonlinear filters is shown. In order to compare the different estimators, they are each derived from various approximations made to a weighted least squares cost function. Specific consideration is given to the differences between a two-step estimator which uses a first and second step state vector that are of equal dimensions and existing filters which make use of a change of variables at each time step to obtain linear measurement equations.

2.1 Review of Suboptimal Nonlinear Estimation

As a prelude to the derivation of the two-step estimator following [Haupt, et al, 1996] and [Kasdin, et al, 1997], several existing suboptimal nonlinear filters are derived starting from various approximations to the same original cost function. This permits a comparison between the assumptions involved in each and their resulting limitations. A statement of the nonlinear estimation problem is to recursively obtain the "best" estimate, $\hat{x}_N$, of the state, $\bar{x}_N$, of a nonlinear system described by the sequence of solutions

$$\bar{x}_{i+1} = \phi(\bar{x}_i, t_{i+1}, t_i) + \bar{w}_i \tag{2.1}$$

to a continuous differential equation at the $N^{TH}$ time step based upon the sequence of discrete measurements $\{\bar{z}_1, ..., \bar{z}_{N-1}, \bar{z}_N\}$ obtained up until time $t_N$. These discrete measurements are given by the nonlinear equation

$$\bar{z}_i = h_i(\bar{x}_i) + \bar{v}_i \tag{2.2}$$
and are \( l \)-dimensional vectors. The dynamic noise, given by \( w_i \), and the
measurement error, given by \( u_i \) are assumed to be zero mean Gaussian random
processes. They are uncorrelated with each other and in time. The complete
statistics for these random variables are described by the following expressions.

\[
\langle u_i u_j^T \rangle = Q_i \delta_{i,j} \\
\langle u_i v_j^* \rangle = R_i \delta_{i,j} \\
\langle u_i v_j^* \rangle = 0 \text{ for all } t_i \text{ and } t_j
\]

(2.3) (2.4) (2.5)

The first step in deriving an estimator is to define a criterion for optimality.
One of the most general is minimization of a weighted least squares cost function of
the form

\[
J = \frac{1}{2} (\bar{z}_0 - \bar{x}_0(-))^T \mathbf{P}_0^{-1} (\bar{z}_0 - \bar{x}_0(-)) + \frac{1}{2} \sum_{i=1}^{N} (\bar{z}_i - h_i(\bar{z}_i))^T \mathbf{R}_i^{-1} (\bar{z}_i - h_i(\bar{z}_i))
\]

(2.6)

with respect to the sequence of \( N \) states \( \{\bar{x}_1, \bar{x}_2, ..., \bar{x}_N\} \) at times \( \{t_1, t_2, ..., t_N\} \) given
the full set of \( N \) observation vectors provided up to and including the time \( t_N \) and
an a priori estimate of the state at time \( t_0, \bar{x}_0(-) \). The sequence of state estimates
is constrained to be noiseless solutions of equation (2.1).

Without any assumptions about the statistical description of the states or
the observations, equation (2.6) is still a valid cost function. The weighting
matrices \( \mathbf{P}_0^{-1} \) and \( \mathbf{R}_i^{-1} \) are usually interpreted as the a priori state covariance
matrix, and the sequence of data covariance matrices, respectively.

For the case of a linear system with linear observations, equation (2.6)
reduces to

\[
J_{KF} = \frac{1}{2} (\bar{z}_0 - \bar{x}_0(-))^T \mathbf{P}_0^{-1} (\bar{z}_0 - \bar{x}_0(-)) + \frac{1}{2} \sum_{i=1}^{N} (\bar{z}_i - H_i\bar{z}_i)^T \mathbf{R}_i^{-1} (\bar{z}_i - H_i\bar{z}_i)
\]

(2.7)

subject to the constraint of linear system dynamics

\[
\bar{x}_i = \Phi_{t_i0} \bar{x}_0
\]

(2.8)
Equation (2.7) can be written in terms of the state at time \( t_0 \) by defining the matrix \( \hat{H}_i = H_i \Phi_{i,0} \). Then equation (2.7) becomes a static cost function.

\[
J_{KF} = \frac{1}{2} (\hat{z}_0 - \hat{x}_0(-))^T P_{0}^{-1} (\hat{z}_0 - \hat{x}_0(-)) + \frac{1}{2} \sum_{i=1}^{N} (\hat{z}_i - \hat{H}_i \hat{z}_0)^T R_i^{-1} (\hat{z}_i - \hat{H}_i \hat{z}_0) \tag{2.9}
\]

This cost function is minimized in terms of the state vector at the time \( t_0 \) which, through equation (2.8) will define the full state time history. This minimization is accomplished by setting the gradient of the cost function equal to zero.

\[
\frac{\partial J}{\partial \hat{x}_0} \bigg|_{\hat{x}_0} = 0 \tag{2.10}
\]

giving the linear least squares solution

\[
\begin{bmatrix}
P_0^{-1} + \sum_{i=1}^{N} \hat{H}_i^T R_i^{-1} \hat{H}_i \\
\end{bmatrix}
\hat{x}_0 = P_0^{-1} \hat{x}_0(-) + \sum_{i=1}^{N} \hat{H}_i^T R_i^{-1} \hat{z}_i \tag{2.11}
\]

This is solved exactly by processing the measurements in sequence starting with the \textit{a priori} values \( P_0(-) \) and \( \hat{x}_0(-) \) and recursively computing \( P_{0,i+1} \) and \( \hat{x}_{0,i+1} \) for \( i = 1, 2, 3, \ldots N \) using equations (2.12) and (2.13).

\[
P_{0,i+1} = (P_{0,i}^{-1} + \hat{H}_{i+1}^T R_{i+1}^{-1} \hat{H}_{i+1})^{-1} \tag{2.12}
\]

\[
\hat{x}_{0,i+1} = \hat{x}_{0,i} + P_{0,i+1} \hat{H}_{i+1}^T R_{i+1}^{-1} (\hat{z}_{i+1} - \hat{H}_{i+1} \hat{x}_{0,i}) \tag{2.13}
\]

As discussed in chapter 1 the notation (+) and (-) is not used for the static estimation problem with the understanding that \( \hat{x}_{0,i+1}(+) = \hat{x}_{0,i+1} \) and \( \hat{x}_{0,i+1}(-) = \hat{x}_{0,i} \).

This recursive state estimate also results in the minimum variance estimate of \( \hat{x} \) and is the classic Kalman filter.

Two important points are illustrated by the linear case. First, note that the set of recursive equations in (2.12) and (2.13) could also be derived by minimizing the local cost function at each individual observation

\[
J_{KF} = \frac{1}{2} (\hat{z}_i - \hat{H}_i \hat{x}_{0,i})^T R_i^{-1} (\hat{z}_i - \hat{H}_i \hat{x}_{0,i}) + \frac{1}{2} (\hat{z}_i - \hat{H}_i \hat{x}_{0,i})^T R_i^{-1} (\hat{z}_i - \hat{H}_i \hat{x}_{0,i}) \tag{2.14}
\]
This sequence of minimizations results in the exact minimization of the cost function in equation (2.9) for the complete measurement sequence. This will not be true in general for the nonlinear problem.

Second, the least squares cost function for the linear problem, with $P_0$ and $R_t$ defined as the a priori state covariance and data covariances, respectively is equal to that obtained by the maximum likelihood criterion if the probability densities are assumed to be independent and Gaussian. This is shown by maximizing the conditional probability density for the augmented system

$$Z_i = \begin{bmatrix} \tilde{z}_i \\ \tilde{x}_0 \end{bmatrix}, \quad H_i = \begin{bmatrix} \tilde{H}_i \\ I \end{bmatrix}$$ (2.15)

The conditional joint probability density of the $N$ augmented observation vectors, conditioned on the state vector $\tilde{x}_0$ is found from the assumption of independence.

$$p(\tilde{x}_N, \tilde{x}_{N-1}, ... \tilde{x}_1 | \tilde{x}_0) = p(\tilde{x}_N | \tilde{x}_0)p(\tilde{x}_{N-1} | \tilde{x}_0)...p(\tilde{x}_1 | \tilde{x}_0)$$ (2.16)

This allows the probability density to be expressed as the product of $N$ Gaussian functions. Multiplying these functions together is equivalent to expressing the exponent as a sum of the exponents of each distribution. This Gaussian probability density, for an $l$-dimensional observation vector, is

$$p(\tilde{x}_N, \tilde{x}_{N-1}, ... \tilde{x}_1 | \tilde{x}_0) = \frac{1}{\sqrt{(2\pi)^N | P|}} \exp \left[ -\frac{1}{2} \sum_{i=1}^{N} (Z_i - H_i \tilde{x}_0)^T \tilde{P}^{-1} (Z_i - H_i \tilde{x}_0) \right]$$ (2.17)

The exponential is maximized by minimizing the same function as in equation (2.9). The assumption of Gaussian densities for the initial a priori error and for the observations results in a state estimate that is also Gaussian for all time as a result of the linear system model.

When the system model is nonlinear, however, many of the aforementioned properties of the linear filter no longer hold true. Specifically, nonlinear estimation problems no longer have the advantage of exact recursive forms for the optimal
solution, nor does the optimal estimate always lend itself to a statistical interpretation as minimum variance and maximum likelihood. The least squares cost function defined in equation (2.6) can be often be minimized iteratively using a batch method which processes the full set of observation vectors at once. This global iteration is repeated until the residuals have decreased below some tolerance. For off line problems in orbit determination and other fields this is an acceptable approach and has been used for many years. For the problem of real time satellite navigation, however, a recursive filter is required. Such practical recursive filters for nonlinear systems are suboptimal. It is desired in this study to utilize a better sub-optimal filter for the dynamics (relative motion between two satellites in elliptical orbits) and observations (intersatellite ranging observations) than conventional methods presently in use.

The statistical interpretation of the estimate obtained by minimizing equation (2.6) is also more difficult than for the linear problem because the nonlinear state propagation does not preserve Gaussian probability densities. Hence, the statistics of the state estimate cannot be described by only two moments; the mean and the covariance matrix. The complete description requires the propagation of the whole probability density function. This problem has been worked by others [Kushner, 1967(a)] [Kushner 1967(b)]. There are some special cases of optimal nonlinear estimation which can be solved exactly with a finite dimensional filter [Daum, 1980]. Unfortunately, the general nonlinear filter state is infinite dimensional [Jazwinski, 1970].

In this study, the development of suboptimal nonlinear filters is done progressively following the classical linear results given earlier. The extended Kalman filter, iterated extended Kalman filter, and a description of specific change of variables in the above filters are subsequently derived from various approximations to the cost function in equation (2.6).
The extended Kalman filter is derived from a linearized cost function assuming that an initial estimate of the true state $\hat{x}_i(-)$ is available at time $t_i$. The linear filter results are applied to determine the (assumed) small deviation from this estimate, $\delta \tilde{x}$ such that $\tilde{x}_i = \hat{x}_i(-) + \delta \tilde{x}_i$. The a priori deviation from the estimate at each time step is zero ($\delta \tilde{x}_i(-) \equiv 0$) by definition and its covariance matrix is $P_i(-)$.

The observation equation is expanded in a Taylor series about $\hat{x}_i(-)$.

$$\tilde{z}_i = h_i(\hat{x}_i(-)) + \left. \frac{\partial h_i}{\partial x} \right|_{x(-)} \delta \tilde{x} + ...$$

(2.18)

Only the first two terms of this expansion are retained and a least squares cost function is defined from the error at the $i^{TH}$ time step following the measurement $z_i$.

$$J_{EKF_i} = \frac{1}{2} \delta \tilde{x}_i^T P_i^{-1} \delta \tilde{x}_i +$$

$$\left[ \tilde{z}_i - h_i(\hat{x}_i(-)) - \left. \frac{\partial h_i}{\partial x} \right|_{x(-)} \delta \tilde{x} \right]^T R_i^{-1} \left[ \tilde{z}_i - h_i(\hat{x}_i(-)) - \left. \frac{\partial h_i}{\partial x} \right|_{x(-)} \delta \tilde{x} \right]$$

(2.19)

Minimization of this cost function by setting

$$\left. \frac{\partial J_{EKF_i}}{\partial (\delta \tilde{x})} \right|_{\delta \tilde{x} = 0} = 0$$

(2.20)

has an exact recursive solution for $\delta \tilde{x}(+)$ using the Kalman filter given in equations (2.12) and (2.13). This can be manipulated into a recursive expression for $\hat{x}_i(\cdot)$ which is the extended Kalman filter (EKF).

The important thing to note about the EKF is that it is derived from a local cost function which uses a linearized approximation of the measurement equation. Hence, it is suboptimal and does not minimize the complete cost function in equation (2.6), or even the true nonlinear local cost function at the $i^{TH}$ time step.

The iterated extended Kalman filter (IEKF) attempts to improve upon the EKF by minimizing the exact local cost function.

$$J_{IEKF_i} = \frac{1}{2} (\tilde{z}_i - \hat{z}_i(-))^T P_i^{-1} (-) (\tilde{z}_i - \hat{z}_i(-)) + \frac{1}{2} (\tilde{z}_i - h_i(\tilde{x}_i))^T R_i^{-1} (\tilde{z}_i - h_i(\tilde{x}_i))$$

(2.21)
The IEKF is derived by minimizing (2.21) through a Gauss-Newton iteration [Bell and Cathey, 1993]. The gradient of $J_{IEKF_i}$, computed using $\hat{x}_i^k$ is

$$\frac{\partial J_{IEKF_i}}{\partial x_i} \bigg|_{\hat{x}_i^k} = -\frac{\partial h}{\partial x} |_{\hat{x}_i^k}^T R_i^{-1} (\tilde{z}_i - h_i(\hat{x}_i^k)) + P_i^{-1}(\hat{x}_i^k - \hat{x}_i(-))$$

(2.22)

and the Gauss approximation to the Hessian matrix is

$$\frac{\partial^2 J_{IEKF_i}}{\partial x^2} \bigg|_{\hat{x}_i^k} = P_i^{-1}(-) + \left[ \frac{\partial h}{\partial x} |_{\hat{x}_i^k}^T R_i^{-1} \frac{\partial h}{\partial x} |_{\hat{x}_i^k} \right]$$

(2.23)

These are substituted into the Gauss-Newton iteration

$$\hat{x}_i^{k+1} = \hat{x}_i^k - \left[ \frac{\partial^2 J_{IEKF_i}}{\partial x^2} \bigg|_{\hat{x}_i^k} \right]^{-1} \frac{\partial J_{IEKF_i}}{\partial x} \bigg|_{\hat{x}_i^k}$$

(2.24)

which, after some manipulation, results in

$$P_i^k = \left[ P_i(-)^{-1} + \frac{\partial h}{\partial x} |_{\hat{x}_i^k}^T R_i^{-1} \frac{\partial h}{\partial x} |_{\hat{x}_i^k} \right]^{-1}$$

(2.25)

$$\hat{x}_i^{k+1} = \hat{x}_i(-) + P_i \frac{\partial h}{\partial x} |_{\hat{x}_i^k}^T R_i^{-1} \left[ \tilde{z}_i - h_i(\hat{x}_i) + \frac{\partial h}{\partial x} |_{\hat{x}_i^k} (\hat{x}_i(-) - \hat{x}_i^k) \right]$$

(2.26)

This is the IEKF update iteration. Equation (2.26) converges to the minimum of (2.21) showing that the IEKF minimizes the nonlinear cost function at each time step generating the a posteriori state and covariance $\hat{x}_i(+) = \hat{x}_i^k$, $P_i(+) = P_i^k$.

However, unlike the linear case, this will not minimize the total cost function given in equation (2.6).

Having identified the approximations made in deriving the EKF and IEKF, the next step is to assess the effects of these approximations on the filter performance for the type of problems considered. In modern satellite orbit determination, the estimation problem is often characterized by frequent measurements of a small set of scalar quantities, such as range and range rate (Doppler). These measurements are nonlinearly related to the state vector. Furthermore, these measurements are often of high precision such that the filter adjusts the gain matrix so as to heavily weight each new measurement. The
dynamics, while inherently nonlinear, can often be approximated as linear owing to
the short time steps between updates. The approximation to the dynamics enters
primarily in the propagation of the state estimate between updates, which is the
approximation that
\[ \langle \tilde{x}(t) \rangle = \langle g(\tilde{x}(t)), t \rangle \approx g(\langle \tilde{x}(t) \rangle), t \] (2.27)
and in the linearization of the covariance matrix propagation.

An early study of the errors which result from linearization
[Denham and Pines, 1966] found that the neglected terms in the measurement
equation can have a significant effect on biases in the filter estimate. In this
analysis, the actual value of the state estimate following a scalar measurement was
expanded to second order and a better approximation to the true \textit{a posteriori} state
covariance was derived. With the assumption of a symmetric probability density
for the state error, third order moments can be eliminated. A second order
expansion of the \textit{a posteriori} state covariance is

\[
P(+) = \langle (\tilde{x} - \hat{x}(+))^T(\tilde{x} - \hat{x}(+)) \rangle = (I - KH)P(-)(I - KH)^T + \]
\[
K \left[ R + \frac{1}{4} \left( \langle (\tilde{x} - \hat{x}(-))^T \frac{\partial^2 h}{\partial x^2} (\tilde{x} - \hat{x}(-)) \rangle \right)^2 \right] K^T \] (2.28)
in which \( K \) is the Kalman gain \( K = (P(-)^{-1} + H^T R^{-1} H)^{-1} H^T R^{-1} \).

Comparison of this equation to the Joseph form of the covariance update
reveals that the actual size of the state error covariance following a measurement
update is larger than that predicted by the Kalman filter update because of the
second partial derivative term. For situations in which the matrices \( R \) and
\( \langle (\tilde{x} - \hat{x}(-))^T \frac{\partial^2 h}{\partial x^2} (\tilde{x} - \hat{x}(-))^2 \rangle \) have elements which are comparable in size,
then the linearization is not very good. The computed covariance under this
situation is smaller following each measurement update and this causes subsequent
measurement updates to be under-weighted giving the filter false confidence in the
state estimates. An interesting result of this analysis is that this problem becomes
worse for more accurate measurements. The important comparison is between for the size of \( R \) and of \( \left| \langle (\bar{x} - \hat{x}(-)) (\partial^2 h / \partial x^3)(\bar{x} - \hat{x}(-)) \rangle \right|^2 \). These terms can be comparable in size even if the linearization of the measurement equation is a good approximation, expressed mathematically as

\[
\left| \frac{\partial h}{\partial x}(\bar{x} - \hat{x}(-)) \right| >> \left| (\bar{x} - \hat{x}(-))^T (\partial^2 h / \partial x^3)(\bar{x} - \hat{x}(-)) \right|
\]  

(2.29)

Denham and Pines went on to show that by performing local iterations, as in the IEKF, the size of this neglected term can be reduced. Following the \( k^{TH} \) iteration, the neglected term is \( \left| \langle (\bar{x} - \hat{x}^k(-))^T (\partial^2 h / \partial x^3)(\bar{x} - \hat{x}^k(-)) \rangle \right|^2 \) Presumably the error is reduced following each iteration, resulting in a subsequent reduction in the size of this term.

Another approach to deal with nonlinear system models is to find, if possible, an invertible coordinate transformation \( \bar{y}_i = f(\bar{x}_i) \) which converts the system of (2.1) into a linear system of equal dimension, in terms of the variable \( \bar{y} \) [Neal, 1968]. Symbolically, applying this coordinate transformation to both sides of equations (2.1) and (2.2) results in the system:

\[
\begin{align*}
\Phi_{i+1}f(\bar{x}_i) + q(i, t_i, w(t_i)) \\
H_i f(\bar{x}_i) + \eta(i, v(t_i))
\end{align*}
\]  

(2.30)

Conventional linear estimation techniques are then applied to the state variable \( \bar{y}_i \), to determine estimates which exactly minimize a linear cost function of the form of (2.7). Two difficulties arise with this method. First, the state errors are no longer zero mean Gaussian distributed random variables because of the nonlinear coordinate transformation. The conditional mean is only approximated by \( \hat{x}_i \approx f^{-1}(\bar{y}_i) \). These results, while an exact solution to the linear optimal estimation problem for \( \bar{y} \), can only be interpreted as a least squares solution. The second step state estimate, \( \hat{x}_i \), is not guaranteed to be a zero mean minimum variance estimate because the probability density created by the inverse transformation \( f^{-1} \) could be
very complicated. The second, and more limiting, difficulty is that it will often be
difficult or impossible to find a function \( f(\tilde{x}) \) which makes both the dynamics and
the measurement equations linear. Furthermore, such a function is not necessarily
invertible in closed form. In that case, an iterative solution for \( \tilde{x} \) is required.

In many cases, as shown by [Denham and Pines, 1966] and others, the
nonlinearities of the measurement equation are the largest source of estimation
bias. For this reason, a nonlinear transformation is sought to convert the
measurement equation alone to a linear expression.

Conceptually, such a filter is applied as follows.

1. Propagate the state and covariance in the nonlinear state space (\( \tilde{x} \)) to the
time \( t_i \) of the \( i^{TH} \) measurement.

\[
\dot{x}_i(\cdot) = \phi(\tilde{x}_{i-1}(\cdot), t_i, t_{i-1})
\]

\[
P_{x_i}(\cdot) = \Phi_{i,i-1}P_{x_{i-1}}(\cdot)\Phi_{i,i-1}^T + \Gamma_iQ_d\Gamma_i^T
\]

2. Transform the state and covariance to the linear state space (\( \tilde{y} \)) using the
change of variables \( \tilde{y}_i = f_i(x_i) \).

\[
\tilde{y}_i(\cdot) = f(\tilde{x}_i(\cdot), t_i)
\]

\[
P_{y_i}(\cdot) \approx \frac{\partial f}{\partial x} \bigg|_{\tilde{x}_i(\cdot)} P_{x_i}(\cdot) \frac{\partial f^T}{\partial x} \bigg|_{\tilde{x}_i(\cdot)}
\]

3. In this system, the observations are exactly described by the linear
equation \( \tilde{z}_i = H_i \tilde{y}_i + \tilde{v}_i \) because \( f() \) has been selected to make

\[
h_i(\tilde{x}_i) = H_if_i(\tilde{x}_i)
\]

Linear estimation methods are then applied to minimize the least squares
cost function at this time step. This generates the \textit{a posteriori} state and
covariance \( \tilde{y}_i(\cdot) \) and \( P_{y_i}(\cdot) \).
(4) Convert the state and covariance back to the nonlinear state space using the inverse transformation

$$\hat{x}_i(+) \approx f_i^{-1}(\hat{y}_i(+))$$  \hspace{1cm} (2.36)

$$P_{\hat{x}_i}(+) \approx \left. \frac{\partial f_i^{-1}}{\partial y} \right|_{\hat{y}_i(+)} P_{\hat{y}_i}(+) \left. \frac{\partial f_i^{-1}}{\partial y} \right|^{T}_{\hat{y}_i(+)}$$  \hspace{1cm} (2.37)

This transformation is assumed to exist and be unique. In cases where a closed form solution for the inverse does not exist, it is necessary to obtain this solution iteratively.

(5) Return to (1) and propagate the updated state and covariance forward to the $k+1^{st}$ time step.

A change of variables similar to this has been successfully applied by others [Aidala and Hammel, 1983], [Tenney, et al., 1977], [Balakrishnan, 1989], [Mehra, 1971]. Although, in general, the use of a nonlinear coordinate transformation can make the probability density very complex function, [Balakrishnan and Speyer, 1985] were able to choose the set of states such that the location of the conditional mode is approximately preserved. Thus the coordinate transformation filter which they developed approximates a maximum likelihood estimator [Mehra 1972].

The single step minimization done on the linear problem in step 3 above, is equivalent to first order to the iterative minimization done by an IEKF applied directly to the nonlinear states [Mehra, 1971]. This is demonstrated by writing the local cost function for a Kalman filter at the $i^{th}$ time step in the transformed system (CTF).

$$J_{CTF_i} = \frac{1}{2}(\hat{y}_i - \hat{y}_i(-))^T P_{\hat{y}_i}^{-1}(-)(\hat{y}_i - \hat{y}_i(-)) + \frac{1}{2}(\bar{z}_i - H_i\hat{y}_i)^T R_i^{-1} (\bar{z}_i - H_i\hat{y}_i)$$  \hspace{1cm} (2.38)

Substitute the expression given in equation (2.33) and (2.35) and expand $f_i(\bar{z}_i)$ in a Taylor series about $\bar{z}_i(-)$. Using equation (2.37) which is a first order
approximation to the second step state covariance matrix, will result in

\[ J_{CTF_i} = \frac{1}{2} (\vec{x}_i - \hat{x}_i(\cdot))^T P_{\vec{z}_i}^{-1}(\cdot)(\vec{x}_i - \hat{x}_i(\cdot)) + \frac{1}{2} (\vec{z}_i - h_i(\vec{x}_i))^T R_i^{-1}(\vec{z}_i - h_i(\vec{x}_i)) \] (2.39)

This is equal to \( J_{IEKF_i} \) in equation (2.21). Hence, the single step Kalman filter applied to a problem in which the states are transformed such that the measurement equation is linear, is expected to result in similar least squares performance to that obtained by the IEKF. The numerical simulations in [Mehra, 1971] demonstrate these results.

### 2.2 Derivation of the Two-Step Filter

The two-step estimator in [Haupt, et al, 1996] and [Kasdin, et al, 1997] provides an improved recursive solution to the state estimation problem involving nonlinear measurements. This method, while still suboptimal for dynamic problems, is expected to provide a better solution to the nonlinear least squares problem. It is derived as the optimal solution to the static nonlinear measurement problem. This solution is then approximated for the dynamic problem. A special case of this filter in which the number of first and second step state are equal is very similar to the coordinate transformation technique described in the previous section. This similarity is examined in the next section of this chapter.

The two-step filter is derived by breaking the nonlinear state estimation problem for an \( m \)-dimensional state into two parts through the definition of an \( n \)-dimensional "first step" state vector. This state vector is of equal or larger dimension than the second step state vector (\( n \geq m \)) and is chosen so as to make the observations linear in the first step states. The nonlinear relationship between the first and second step states in cases where \( n > m \) is over-determined and must be solved for iteratively.

First the nonlinear static problem is considered. The cost function of (2.6)
without an a priori state estimate reduces to

\[ J_{\text{STATIC}} = \frac{1}{2} \sum_{i=1}^{N} (\vec{z}_i - h_i(\vec{z}_i))^T R_i^{-1} (\vec{z}_i - h_i(\vec{z}_i)) \]  

(2.40)

Assume a separable nonlinear measurement equation

\[ h_i(\vec{x}_{0,i}) = H_i f(\vec{x}_0) \]  

(2.41)

The time variation of the measurement equation is thus contained entirely in the \( l \)
by \( n \) observation matrix \( H_i \). The function \( f(\vec{x}_0) \) is a \( n \) dimensional vector function of the constant state vector \( \vec{x}_0 \), which defines the first step state vector \( \vec{y}_0 \). The observations are now processed as if they result from a linear system

\[ \vec{z}_i = H_i \vec{y}_0 + \vec{e}_i \]  

(2.42)

using the standard methods of linear filtering. This minimizes the cost function defined for the (augmented) system of first step states

\[
\begin{bmatrix}
\vec{z}_1 \\
\vec{z}_2 \\
\vdots \\
\vec{z}_N
\end{bmatrix}
= \begin{bmatrix}
H_1 \\
H_2 \\
\vdots \\
H_N
\end{bmatrix}
\begin{bmatrix}
R_1 & 0 & \cdots & 0 \\
0 & R_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & R_N
\end{bmatrix}
\]  

(2.43)

\[ J_y = \frac{1}{2} (Z - H_0 \vec{y}_0)^T R^{-1} (Z - H_0 \vec{y}_0) \]  

(2.44)

The cost function in (2.40) can be minimized by writing it in terms of the separated measurement equation. First minimize \( J_y \) in (2.44) (through the standard linear methods) with respect to the estimates of the first step states and then subsequently minimize the following least squares cost function for the second step states using the first step states as observations.

\[ J_x = \frac{1}{2} (\vec{y}_0 - f(\vec{x}_0))^T P_{y}^{-1} (\vec{y}_0 - f(\vec{x}_0)) \]  

(2.45)
Taking the first derivative of (2.45) gives
\[(\tilde{y}_0 - f(\tilde{x}_0))^T R_y^{-1} \frac{\partial f}{\partial x} = 0\] (2.46)

Minimizing (2.44) is done by setting \(\partial J_y/\partial y = 0\) which can be rearranged as
\[\tilde{z}^T \tilde{R}^{-1} \tilde{H} = \tilde{y}_0^T \tilde{H}^T \tilde{R}^{-1} \tilde{H}\] (2.47)

Substituting this into (2.46) results in
\[\sum_{i=1}^{N} (\tilde{z}_i - h_i(\tilde{x}_0)) R_i^{-1} \frac{\partial h_i}{\partial x} = 0\] (2.48)

The covariance matrix \(P_y\) for the solution in (2.47) is \(P_y^{-1} = \tilde{H}^T \tilde{R}^{-1} \tilde{H}\). This substitution, as well as equation (2.41) have been made. With these substitution, it is shown that the result in equation (2.48) is identical to what would be obtained by computing \(\partial J_{STATIC}/\partial x\) from equation (2.40) directly. Hence, sequential minimization, first with respect to \(\tilde{y}_0\) and then with respect to \(\tilde{x}_0\) gives the exact same results as minimization of the full static cost function \(J_{STATIC}\) in equation (2.40).

It is the ability to separate the time variation of the observation equation from the nonlinearity that makes the two-step static estimator optimal. As long as the transformation is chosen in such a way as to make the time variation appear only in a linear manner any set of states can be used and the subsequent least squares minimization in equation (2.45) will guarantee minimization of the full cost function (2.40).

In the optimal static form, unfortunately, the two-step estimator is not useful for the orbital navigation problem considered in this study. In [Haupt, et al, 1996] the two-step estimator is extended to become a sub-optimal filter for dynamic problems with nonlinear measurements. A set of \(n\) first step states is chosen with the same requirements as in the static case, namely that the observation can be written exactly as a linear function of these states. The
difference in the dynamic case is that the nonlinear portion of the measurement
equation now also contains a time variation, either directly or through the variation
of the first step states.

\[ \tilde{y}_i = f_i(\tilde{x}_i) \]  

(2.49)

The extension of the static filter equations to the dynamic case is made by
an approximation to the first step state propagation at each time step as a linear
term with a similar structure to process noise. This is derived from the identity

\[ \tilde{y}_i = \tilde{y}_{i-1} + f_i(\tilde{x}_i) - f_{i-1}(\tilde{x}_{i-1}) \] 

(2.50)

The function \( f() \) is expanded about the a priori and a posteriori second step state
estimates then the expected value is taken. Terms to first order are kept which
results in the approximation.

\[ \hat{y}_i(\cdot) \approx \hat{y}_{i-1}(\cdot) + f_i(\hat{x}_i(\cdot)) - f_{i-1}(\hat{x}_{i-1}(\cdot)) \] 

(2.51)

The covariance for this update of the first step states is obtained in the
usual manner, by subtracting equations (2.51) from (2.50) and computing the
expected value.

\[ P_{y_i}(\cdot) = \langle (\tilde{y}_i - \hat{y}_i(\cdot))(\tilde{y}_i - \hat{y}_i(\cdot))^T \rangle \] 

(2.52)

When the function \( f() \) is approximated to first order as in (2.51) then equation
(2.52) results in.

\[ P_{y_{i+1}}(\cdot) = P_{y_i}(\cdot) + \left. \frac{\partial f_{i+1}}{\partial x} \right|_{\tilde{x}_{i+1}(\cdot)} P_{x_{i+1}}(\cdot) \left. \frac{\partial f_i}{\partial x} \right|_{\hat{x}_{i}(\cdot)} - \left. \frac{\partial f_{i+1}}{\partial x} \right|_{\tilde{x}_{i+1}(\cdot)} P_{x_i}(\cdot) \left. \frac{\partial f_i}{\partial x} \right|_{\hat{x}_{i}(\cdot)}^T \] 

(2.53)

This generates the a priori covariance matrix at the \( i + 1^{st} \) time step by
adjusting the a posteriori covariance from the \( i^{th} \) step by the difference between
two positive semidefinite matrices. It is not clear that equation (2.53) is guaranteed
to produce a positive-definite covariance matrix, owing to the negative sign on the
third term. In fact, this equation can occasionally result in covariance matrices
with very low eigenvalues. Sometimes these small eigenvalues become numerically negative which represents a physically meaningless condition and occasionally results in failure of the iterative second step solution. This condition and some suggested modifications to the two-step filter are studied extensively in chapter 3.

Mechanization of the two-step filter is given in the references [Haupt, et al, 1996] and [Kasdin, et al, 1997] as follows: 1) Perform a standard linear measurement update of the first step state estimate and covariance based on the observations. 2) Compute the second step state estimates by a nonlinear estimation process such as the Gauss-Newton or Lavenberg-Marquardt [Press, et al., 1994] methods and update the second step covariance matrix by a first order approximation. 3) Propagate the second step states and covariance matrix forward to the time of the next measurement using the state transition matrix or numerical integration as required. 4) Propagate the first step states and covariance matrix to the time of the next measurement using the a posteriori first and second step covariance matrices from the last measurement update and the propagated (a priori) second step covariance matrix for the upcoming measurement epoch.

The update of the second step covariance from the a posteriori first step covariance following the iterative update of the of the second step state (step number 2 above) is obtained by using the Gauss approximation to the Hessian matrix.

\[ p_{\theta_i}^{-1}(+) = \frac{\partial f}{\partial x} \bigg|_{\hat{x}_i(+)}^T \ p_{\theta_i}^{-1}(+) \ \frac{\partial f}{\partial x} \bigg|_{\hat{x}_i(+)} \]  \hspace{1cm} (2.54)

Choosing the set of first step states is left to engineering judgment with the only constraint being that the set results in a linear observation equation and that the second step states are observable given the first step states. For any nonlinear measurement equation, such a set can always be chosen. One direct method of creating the first step state vector is to augment the second step state vector with
the measurement equation [Haupt, et al, 1996].

\[
\begin{bmatrix}
\tilde{y} \\
\tilde{x}
\end{bmatrix} = \begin{bmatrix}
h(\tilde{x}) \\
I
\end{bmatrix} \quad H = \begin{bmatrix} I & 0 \end{bmatrix}
\]

(2.55)

This procedure for selecting first step states is more general than a coordinate transformation method. Any measurement equation can be placed into the state augmented form of equation (2.55). It is not always possible to find a coordinate transformation which produces a linear measurement equation.

2.3 Comparison of the $n = m$ Two-Step Filter to the Coordinate-Transformation-Based Kalman Filter

For the special case of a two-step filter with an equal number of first and second step states, equations (2.54) and (2.53) appear very similar to equations (2.37) and (2.34) for a coordinate transformation based Kalman filter. The update of the second step states in both filters is the same. For this reason, one might expect that the two filters would have similar performance. However, the coordinate transformation filter was shown to have a cost function which was equivalent, to first order, to the IEKF cost function. A two-step filter is expected to have a lower mean square error than the IEKF.

To explain why the two-step filter is expected to produce a better estimate than the coordinate transform filter using the same set of first step, or “transformed” states, consider that equations (2.53) and (2.34) are both first order approximations to the propagation of $P_y$. Computation of the exact first step state covariance requires evaluation of the expected value of the first step squared error. While this could be done by numerical integration or a Monte-Carlo method to any desired tolerance, it is impractical to do on-line at each time step. These direct methods are, however, useful for initializing the filter (as explained in chapter 5).

The error in the first order expansion of $P_y$ is comprised of third and
higher moments of the first step state variable, $\tilde{x}_i$. Symbolically all of these terms are combined together into a single error, $G_i$.

$$P_{yi} = \frac{\partial f}{\partial x} \bigg|_{\tilde{x}_i} P_{xi} \frac{\partial f^T}{\partial x} \bigg|_{\tilde{x}_i} + G_i$$  \hspace{1cm} (2.56)

The error in $P_y$ made at each time step in the coordinate transformation based Kalman filter is equal to $G_i$. This is because a new first order approximation to $P_y$ is made at each time step. In the two-step filter, however, it is only the change in $P_y$ between the $i^{th}$ and the $i + 1^{st}$ time steps which is approximated. Hence the error made at each time step in the two-step filter is $G_{i+1}(-) - G_i(+)$. If measurements are made at closely spaced intervals then the elements of $G_{i+1}(-)$ and $G_i(+) \text{ are not very different and } G_{i+1}(-) - G_i(+) \text{ is less than the magnitude of the elements in } G_i$. This conclusion, however, assumes that an accurate initial $P_{y0}$ is used. Results from numerical simulations of the orbital relative navigation problem are given in chapter 6 demonstrating that the two-step filter with $n = m$ reduces a large initial error faster than a coordinate transformation based filter using the same set of states. This filter gave similar performance to a two-step filter with $n > m$. 

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CHAPTER 3

NUMERICAL TEST FOR ILL-CONDITIONED $P_y$

In this chapter it is shown that the two-step estimator as described in [Haupt, et al, 1996] and [Kasdin, et al, 1997] can generate a first step covariance matrix, $P_y$, with a very small eigenvalue. This presents a problem for the filter because the iterative solution is not numerically sound if such an ill-conditioned $P_y$ is used in the formation of a cost function. Furthermore, the practical implementation of the filter algorithm has to be done on a computer with finite numerical precision. This opens the possibility of a very small eigenvalue becoming numerically negative. In many cases this causes the iterative solution to catastrophically fail as the result of a meaningless negative cost function. The mathematical explanation for this anomaly is shown and a test is derived to identify those points in the state space in which the minimum eigenvalue drops below a specified numerical tolerance. Several simple modifications to the existing two-step filter algorithm are suggested which reduce the effects of this anomaly on filter performance and eliminate the possibility of it causing an outright failure as the result of a negative eigenvalue.

3.1 Analysis Of The Problem

The form of equation (2.53) involves the subtraction of two positive semidefinite matrices and as such, it is not guaranteed to always generate a positive definite covariance matrix $P_y$. The subsequent least squares estimate of the second step states, however, requires that this weighting matrix be positive definite. When simulations of the orbital navigation problem are performed, the first step
covariance matrix occasionally has very low eigenvalues at distinct points in the state time history. At times, it is even observed that small negative eigenvalues are also generated, possibly as a result of machine precision limitations.

Unfortunately, even the use of a UD covariance factorization, as given in [Kasdin, et al, 1997], which does help to reduce the effect of an ill-conditioned $P_y$, does not eliminate the existence of very low and occasionally even negative eigenvalues. Equation (2.53), when factored, still involves the subtraction of two matrices. The subtraction appears as a negative sign on one of the diagonal submatrices used to form the $D$ matrix. No square root form of this equation which guarantees a non-negative definite $P_y$ is known.

An example of this anomaly and its effect on filter operation is illustrated in figure 3.1. The simulation in this figure is a case of processing range rate measurements to estimate relative position and velocity between two orbiting vehicles. The first step state vector is defined by augmenting the second step state vector with the measurement equation as in equation (2.55). Hence the dimension of the first step state space is $n = 7$ and the dimension of the second step state space is $m = 6$. The batch least squares minimization of the second step cost function in equation (2.45) is limited to a maximum of 2000 iterations. Specific details of the truth model and filter states for the relative orbit motion problem are given in chapters 4 and 5.

As figure 3.1(a) shows, near $f_p \approx 1.1$ one of the eigenvalues of $P_y(-)$ becomes very small. The ratio of the smallest eigenvalues to the largest eigenvalues is on the order of $10^{-16}$ to $10^{-17}$. The result of this low eigenvalue on the filter is illustrated in the subsequent three plots. Figure 3.1(b) shows that the sign of the smallest eigenvalue of $P_y(-)$ becomes negative at some points as a result of the machine precision. The computer on which these simulations were performed had a floating point relative accuracy of $\epsilon = 2.2 \times 10^{-16}$. This is defined as the smallest
Figure 3.1. Example of Non-Positive Definite $P_y$ in the Relative Orbital Navigation Problem. a.) Time History of the Eigenvalues of $P_y$. b.) Sign of the Eigenvalues of $P_y$ Showing Negative Values. c.) Iterations Required for Second Step Solution to Converge. d.) A Posteriori State Error.
floating point number for which $1 + \epsilon \neq 1$ [The Mathworks, Inc. 1992].

As a consequence of the negative eigenvalue in $P_y$, the iterative least squares solution for the second step state began to diverge. Recall that the inverse of $P_y$ is used as the weighing matrix on the second step cost function and its gradient. The smallest eigenvalue of $P_y$ corresponds to the direction in which the iterative minimization will place the largest weight. If this eigenvalue becomes negative, then there is a possibility that the cost function and hence its gradient will also become negative. A negative cost function will cause the projected gradient update to be performed in the opposite direction. In some cases it is thought that this is the actual mechanism which leads to failure of the second step iterative minimization.

The divergence of the orbital example is shown in figure 3.1(c). Note that the number of iterations reaches the maximum of 2000. Ultimately, the filter state estimate diverges as is shown in figure 3.1(d). The exact behavior of the second step minimization once this point is reached is highly variable and depends on other parameters of the filter such as the test criteria used for stopping the iterations, the tolerance on the stopping criterion, the iterative algorithm, and the complexity of the problem. In this example the second step cost function was monitored for each update and the state update was taken once the cost function decreased by less than 0.01 from one iteration to the next.

One interesting feature of this anomaly is that it occurs very near a distinct point in the orbit, as opposed to being a gradual degradation of the condition number of $P_y$. This point becomes "sharper" as the sampling interval between discrete points becomes smaller. This consideration indicates that it is not a manifestation of a numerical problem alone which would not exist in a mathematically perfect filter. Furthermore, the location of this failure, while at a very distinct point on a given run, occurs in different locations for different
observation sets chosen from the ensemble started at exactly the same initial conditions. This consideration indicates that the anomaly is not the result of a singularity in the orbital geometry specific to this problem.

The analysis of the cause of ill-conditioned first step covariances is broken into three steps. First, the filter equations are expressed in terms of a new matrix, $C$, which is a function of the two state covariance matrices, $P_x$ and $P_y$. Second, two properties are found for the column space of $C$ which allow this space to be represented by a reduced number of constant basis vectors. Finally, a numerical test is derived using this set of basis vectors and the reference or predicted state trajectory which identifies points in which $P_y$ may become ill-conditioned.

3.1.1 Formulation in Terms of the Matrix $C$

The covariance propagation equations for $P_x$ and $P_y$ in the two-step estimator described in the previous section are expressed in terms of another matrix, $C$, defined at each time step for the a priori and a posteriori conditions.

$$C_i(\pm) \equiv P_y(\pm) - \left. \frac{\partial f}{\partial x} \right|_{\dot{x}_i(\pm)} P_x(\pm) \left. \frac{\partial f}{\partial x} \right|_{\dot{x}_i(\pm)}^T$$

(3.1)

With this substitution, equation (2.53) reduces to

$$C_{i+1}(-) = C_i(+)$$

(3.2)

Next, two assumptions are made about the evolution of the first and second step covariance matrices after any large initial state error has been removed.

Assumption 1: $P_{z_i}^{-1}(-)$ is approximated as:

$$P_{z_i}^{-1}(-) \approx \left. \frac{\partial f}{\partial x} \right|_{\dot{x}_i(-)}^T P_{y_i}^{-1}(-) \left. \frac{\partial f}{\partial x} \right|_{\dot{x}_i(-)}$$

(3.3)

This equation is found by taking equation (2.54), defined for the a posteriori conditions, and applying it to the a posteriori conditions.
Assumption 2: The partial derivative matrix is approximately constant across measurement updates.

\[
\frac{\partial f}{\partial x}|_{\hat{x}_{i}(+)} \approx \frac{\partial f}{\partial x}|_{\hat{x}_{i}(-)} \approx \frac{\partial f}{\partial x}|_{\hat{x}_{i}}
\]  

(3.4)

Computing this partial matrix from the true, reference, or predicted state are all assumed to be equivalent. The state estimate is of interest in this analysis only as it enters the two-step estimator covariance prediction through the partial derivative matrix, \(\partial f/\partial x\). No reference will be made to which side of the measurement the partial derivative matrix is computed on.

These two assumptions, taken together, amount to approximating the second step covariance matrix propagation along a nominal trajectory as equal to that of a linearized Kalman filter which processes measurements and directly updates the second step states. This is shown as follows; The first step covariance update in the two-step filter is given by

\[
P_{y_{i}}^{-1}(+) = P_{y_{i}}^{-1}(-) + H^{T}R_{i}^{-1}H
\]  

(3.5)

Premultiply this \(n\) by \(n\) matrix equation by \((\partial f/\partial x)|_{i}^{T}\) and postmultiply it by \((\partial f/\partial x)|_{i}\). Then use (3.3) and (3.4) to reduce the resulting \(m\) by \(m\) matrix expression to the form of a second step state covariance update

\[
P_{x_{i}}^{-1}(+) = P_{x_{i}}^{-1}(-) + H_{x_{i}}^{T}R_{i}^{-1}H_{x_{i}}
\]  

(3.6)

The measurement matrix is \(H_{x_{i}} = H(\partial f/\partial x)|_{i}\). Equation (3.6) is recognized as the covariance update in an linearized Kalman filter processing measurements \(\hat{x}_{i}\) and updating the second step state directly from them. The propagation of the second step states between measurements is the same for the two-step filter as the linearized filter. Therefore, these two assumptions can be interpreted as the assumption that the evolution of the covariance matrices in two-step filter is the same as that for a linearized Kalman filter applied to the same problem. This was found to be true once the large initial state errors were reduced.
3.1.2 Properties of the C Matrix

Here, two important properties of the C matrix are derived which greatly simplify its propagation and ultimate use in the test for ill-conditioned $P_y$, are derived. Although conditions under which $P_y$ becomes ill-conditioned are of interest here, it is necessary to assume that the matrix $P_y^{-1}$ exists. These two ideas are reconciled by interpreting this analysis as an inquiry into where $P_y$ becomes numerically lower rank, assuming that it will always have an inverse analytically. The numerical definition of matrix rank is the number of singular values having magnitudes larger than some tolerance. When this definition is applied to simulations used later to demonstrate properties of C (which, as will be shown, require forming $P_y^{-1}$) the tolerance on the singular values is set much lower than that which is subsequently be used to indicate when $P_y$ drops rank.

**First Property**: The Rank of C is $n - m$. This property is obtained by arranging equation (3.1) as an expression for $P_{yi}(+)$ given $C_i(+) + P_{zi}(+)$

$$P_{yi}(+) = C_i(+) + \frac{\partial f}{\partial x_i} P_{zi}(+) \frac{\partial f}{\partial x_i}^T$$

(3.7)

and noting that the term

$$\frac{\partial f}{\partial x_i} P_{zi}(+) \frac{\partial f}{\partial x_i}^T$$

is of rank $m$ (assuming $P_{zi}(+)$ is full rank). In order for $P_{yi}(+)$ to be full rank, $C_i(+) + P_{zi}(+)$ must contain at least $n - m$ column vectors which are linearly independent of the column vectors of (3.8). Hence, the requirement

$$\text{rank}(C_i(+)) \geq n - m$$

(3.9)

Post-multiplying equation (3.1) by the matrix $P_{yi}^{-1}(+) \frac{\partial f}{\partial x_i}$

$$C_i(+) P_{yi}^{-1}(+) \frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_i} P_{zi}(+) \frac{\partial f}{\partial x_i}^T P_{yi}^{-1}(+) \frac{\partial f}{\partial x_i}$$

(3.10)

and using equation (2.54) results in

$$C_i(+) P_{yi}^{-1}(+) \frac{\partial f}{\partial x_i} = 0$$

(3.11)
Equation (3.11) indicates that the \( m \) columns of \( P_{yi}^{-1}(+) \partial f / \partial x_i \) are in the nullspace of \( C_i(+) \). If the first step states are independent of each other at each time step, then the matrix \( \partial f / \partial x_i \) is of rank \( m \). This would always be true in the suggested practice [Haupt, et al, 1996] of defining the first step states as the set of second step states augmented by the nonlinear measurement equations. In this case \( m \) rows of the partial derivative matrix form an \( m \) by \( m \) identity.

The rank of the matrix product \( P_{yi}^{-1}(+) \partial f / \partial x_i \) is therefore less than or equal to \( m \) \( (P_{yi}^{-1}(+) \) must be full rank \( ) \). If the rank of this product was less than \( m \), however, then there would exist some vector \( \vec{a} \neq 0 \) such that

\[
P_{yi}^{-1}(+) \partial f / \partial x_i \vec{a} = 0.
\]

Post-multiplying the partial derivative matrix, which must have \( m \) linearly independent columns, by a nonzero vector, \( \vec{a} \), will give another non-zero vector, \( \vec{b} \), \( (\partial f / \partial x)_i \vec{a} = \vec{b} \neq 0 \). This results in the contradiction

\[
P_{yi}^{-1}(+) \vec{b} = 0.
\]

Therefore, the matrix product \( P_{yi}^{-1}(+) \partial f / \partial x_i \) must have a rank equal to \( m \). It is therefore concluded that \( C_i(+) \) also has at least \( m \) linearly independent null vectors and hence the dimension of its nullspace is at least \( m \).

\[
null(C_i(+)) \geq m \tag{3.12}
\]

Combining equations (3.9) and (3.12) and using the fundamental relationship for any \( n \) by \( n \) matrix [Strang, 1976]

\[
rank(C_i(+)) + null(C_i(+)) = n \tag{3.13}
\]

results in the conclusion

\[
rank(C_i(+)) = n - m \tag{3.14}
\]

and, consequently

\[
null(C_i(+)) = m \tag{3.15}
\]

From equation (3.2), the nullspace and column space of \( C_{i+1}(-) \) are identical to those same spaces defined for \( C_i(+) \). This rank property would
consequently apply to the a priori condition as well.

\[ \text{rank}(C_{i+1}(-)) = n - m \]  \hspace{1cm} (3.16)

**Second Property:** The Column Space of \( C \) is Fixed in \( \mathbb{R}^n \) for a filter with \( l \leq m \) after the large initial state error has been removed. Consider that the nullspace and row space of a matrix are orthogonal complements, i.e., every vector in one must be orthogonal to every vector in the other. Therefore, if it can be shown that the nullspace of \( C \) remains fixed, then the row space of \( C \) must also be fixed. The row space and column space of \( C \) are equivalent because it is a symmetric matrix (defined as the sum of two symmetric matrices). This analysis will therefore focus on showing that the nullspace of \( C \) remains constant.

To simplify the notation, the matrices \( N_i(+) \) and \( N_i(-) \) will be defined for the products

\[ N_i(\pm) \equiv P_{yi}^{-1}(\pm) \frac{\partial f}{\partial x_i} \]  \hspace{1cm} (3.17)

The statement that the nullspace of \( C \) remains the same between measurement updates (equation (3.2)) is equivalent to the statement that the columns of \( N_{i+1}(-) \) span the same space as the columns of \( N_i(+) \). This can be written as

\[ N_{i+1}(-) = N_i(+) A_{i+1} \]  \hspace{1cm} (3.18)

in which \( A_{i+1} \) is a set of coefficients expressing each column vector of \( N_{i+1}(-) \) as a linear combination of the column vectors of \( N_i(+) \).

The update of the first step covariance at the \( i + 1 \)st time step using the standard Kalman filter is

\[ P_{yi+1}^{-1}(+) = P_{yi+1}^{-1}(-) + H^T R_{i+1}^{-1} H \]  \hspace{1cm} (3.19)

Post-multiplying by \( (\partial f/\partial x)|_{i+1} \), expressing this in terms of the \( N_{i+1}(\pm) \) matrix
defined in (3.17) and substituting in (3.18) results in

\[ N_{i+1}(+) = N_i(+)A_{i+1} + H^T R_{i+1}^{-1} H \frac{\partial f}{\partial x}|_{i+1} \]  \hspace{1cm} (3.20)

Consider first the general condition where \( H^T \) does not initially lie in the column space of \( N_i(+) \). The \( H \) matrix is assumed to be constant because in principle, for the two-step filter, all time variation in the measurement equation can be incorporated in the definition of the first step states. As each new measurement that is processed, a new term of the form \( H^T R^{-1} H (\partial f / \partial x) \) is added to \( N_i(+) \).

The column vectors of this term, which are in the column space of \( H^T \), are the only modification that the filter can make to the nullspace of \( C \). The only directions in which the columns of \( N \) can be changed is in the directions spanned by the columns of \( H^T \). One would expect that as \( i \) increases, some \( l \) dimensional subspace of the column space of \( N_i(\pm) \) would approach the column space of \( H^T \). Although this is not a rigorous proof, numerical simulations run on models with \( l \leq m \) all have produced a \( C \) with a column space that stays fixed after the initial state transients are reduced and a nullspace which contains the column space of \( H^T \).

Once the the column space of \( N_i(+) \) contains the column space of \( H^T \) as a subspace then the columns of \( H^T \) can be used as \( l \) of the basis vectors for the column space of \( N_i(+) \). The other basis vectors are defined as the columns of an \( n \) by \( m - l \) matrix \( \tilde{N} \). The matrix \( N_i(+) \) can therefore be expressed as the linear combination

\[ N_i(+) = \tilde{N} B_i + H^T D_i \]  \hspace{1cm} (3.21)

Substituting this into (3.20) gives

\[ N_{i+1}(+) = \tilde{N} B_{i+1} + H^T \left[ D_i A_{i+1} + R_{i+1}^{-1} H \frac{\partial f}{\partial x}|_{i+1} \right] \]  \hspace{1cm} (3.22)

This shows that the same basis \{ \( \tilde{N}, H^T \) \} spans the column space of both \( N_{i+1}(+) \) and \( N_i(+) \) and consequently spans the nullspace of both \( C_{i+1}(+) \) and \( C_i(+) \).
Therefore, if \( l \leq m \) and \( H^T \) is in the nullspace of \( C \), it has been shown the column space of \( C \) remains fixed in \( R^n \).

Three special cases will now also be considered. First, if \( l = m \) and the filter is started with little \textit{a priori} state knowledge (\( P_{y_0}^{-1} \) very small), the matrix \( N_0(+) \) can be approximated as

\[
N_0(+) \approx H^T R_0^{-1} H \frac{\partial f}{\partial x} \bigg|_0
\]  

(3.23)

for the initial time steps, because the magnitudes of the column vectors of \( H^T R_0^{-1} H \) are much larger than those of the columns of \( P_{y_0}^{-1}(+) \). For this special case, the columns of \( H^T \) span the nullspace of \( C \), and \( \tilde{N} \) would be the empty set.

Second, in the case of \( l > n \), the same arguments used above can be used to show that the columns of \( H^T \) are the only modifications which are possible to the column space of \( C \), but there are too many column vectors in \( H^T \) to form a basis. For this reason, it will not generally be true that the column space of \( C \) stays fixed for \( l > m \). The span of the columns of the matrix product

\[
H^T R_1^{-1} H \frac{\partial f}{\partial x} \bigg|_1
\]  

(3.24)

does not stay fixed because of the partial derivative factor. It is a changing \( m \)-dimensional subspace of a fixed \( l \)-dimensional space. The basis of the nullspace of \( C \) would be determined by the accumulation of terms like (3.24) from all previous time steps. This case would be rare in actual practice. Most estimation problems do not involve an observation vector which has a larger dimension than the state vector.

Finally, for the case in which the number of first and second step states are equal (\( n = m \)), then the first property of the \( C \) matrix states that its rank is zero. This is seen to be trivially so because equation (2.54), when substituted into equation (3.1) cancels out the term \( P_{y_1}(+) \). Equation (3.1) then reduces to the identity \( C(+) = 0 \), which is curiously a rank 0 matrix. The test condition for
ill-conditioned $P_y$ will later be shown to apply to this case as well, if in a trivial manner.

Interestingly, when the last substitutions are made, the two-step filter reduces to a coordinate transformation based Kalman filter as discussed in chapter 2. This is consistent with the assumption in the present chapter is assumed that the large initial error has been removed and that the covariance update behaves as a linearized filter.

It is only the space spanned by the columns of $C$ which is of interest, not the actual elements of the $C$ matrix. This space has been demonstrated to remain fixed in $R^n$ when $l \leq m$. Thus, the notation \{\$\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_{n-m}\}$ will be used to indicate the basis for the column space of $C$ without any reference to a specific time step or a priori or a posteriori state.

It should be emphasized that these results are independent of how the second state covariance is propagated between measurements. Including process noise does not affect the possibility of generating the ill-conditioned covariance matrices. It may, however, change the specific location of these anomalies by generating a different $C$ matrix.

### 3.1.3 Test for Ill-conditioned First Step Covariance

The $C$ matrix is now used to derive a test to predict the location of points in which the $P_y$ matrix may become ill-conditioned. It must be assumed that $P_y^{-1}$ still exists and that the filter will never actually generate a singular $P_y$. The test which is desired will be one which sets a numerical tolerance on how close to singular we allow $P_y$ to become. Such a rank test can be derived by starting with (3.7).

As mentioned before, equation (3.7) expresses a matrix which must be of rank $n$ as the sum of a matrix of rank $n - m$ and one of rank $m$. This would be...
possible as long as the \( n - m \) basis vectors \( \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_{n-m} \} \) are linearly independent of the column space of \( (\partial f/\partial x)P_x(\partial f/\partial x)^T \). This has already been shown when deriving the first property of \( C \). We now consider cases where at least one basis vector of \( C \) becomes close to being linearly dependent on the column space of \( (\partial f/\partial x)P_x(\partial f/\partial x)^T \). To simplify this further and remove the time-dependent and filter-state-dependent covariance \( P_x \), consider that the column space of \( (\partial f/\partial x)P_x(\partial f/\partial x)^T \) is spanned by the \( m \) columns vectors \( \{\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_m \} \). This leads to a test for conditions under which the set of \( n - m \) vectors \( \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_{n-m} \} \) become close to being linearly dependent upon the set of \( m \) column vectors \( \{\partial f/\partial x_1, \partial f/\partial x_2, \ldots, \partial f/\partial x_m \} \). Recall from the assumption in equation (3.4) these partial derivatives can be computed along a reference trajectory once the space \( \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_{n-m} \} \) has converged to a fixed orientation.

This can be summarized by a numerical rank test of an \( n \) by \( n \) matrix.

\[
\text{rank} \left( \begin{bmatrix} \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_{n-m}, & \frac{\partial f}{\partial x_1}, & \frac{\partial f}{\partial x_2}, & \ldots, & \frac{\partial f}{\partial x_m} \end{bmatrix}, \epsilon \right) < n \tag{3.25}
\]

for some tolerance \( \epsilon \). Points for which the condition described in equation (3.25) is true are the locations where an ill-conditioned first step covariance matrix occurs.

The numerical rank of an \( n \) by \( n \) matrix is defined as the number of singular values greater than \( \epsilon \) [Golub and Van Loan, 1989]. The span of \( \{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_{n-m} \} \) is fixed (property 2) and the partial derivative matrix is computed along a reference trajectory (assumption 1). Therefore, the test defined in (3.25) could conceivably be applied to points ahead of the present filter state to identify future trajectory points in which generating an ill-conditioned covariance matrix is possible.

For the special case of \( n = m \), it was shown that the \( C \) matrix is identically zero. In this case the rank test in equation (3.25) reduces to simply a test on the rank of the partial derivative matrix. \( \text{rank}[\partial f/\partial x, \epsilon] < n \). In this case, the only possible cause of ill-conditioned first step covariance matrices is a near singular
partial derivative matrix which would still be identified through the rank test. In the case of $n = m$ the first step states are no longer under determined, hence it is not possible for there to be a linear combination of first step states with very low uncertainty except under conditions of a singular transformation between first and second step states. This is a fundamental difference between the $n = m$ and $n > m$ cases. In the former, a rank deficient matrix can only result from a singular (first order approximation to) coordinate transformation whereas in the latter this condition represents some combination of the first step states which is redundant.

### 3.1.4 Geometric Interpretation

These concepts are visualized geometrically using a system of two second step states and three first step states ($n = 3$ and $m = 2$) as shown in Figure 3.2. The covariance matrix $P_x$ is represented by the two-dimensional error ellipse in the $x_1 - x_2$ plane. The effect of (3.8) is to rotate and scale that ellipse to a new orientation in the $y_1 - y_2 - y_3$ space. The column space of $\partial f / \partial x$ defines the plane of that ellipse. The first property of the $C$ matrix is that its rank is 1, hence its column space is a single line in the $y_1 - y_2 - y_3$ space as shown in Figure 3.2. The effect of equation (3.7) is to add the additional basis vector from $C$ to the column space of $P_y$.

The second property of $C$ states that this vector approaches a fixed orientation in $y_1 - y_2 - y_3$ space. Figure 3.3 illustrates this evolution of the nullspace and column space of $C$. The plane representing the nullspace of $C$ begins at a location determined by the initial conditions on $P_x$ and $P_y$. As this space is modified by the addition of members of the vector space defined by $\text{span}(H^T)$, the nullspace rotates so that the $H^T$ vector eventually lies in that plane. The component of the nullspace of $C$ which lies in the plane orthogonal to the $H^T$ column space cannot be changed by the filter update.
Figure 3.2: Geometric Interpretation of the Column Space of $P_y$
Figure 3.3: Evolution of the Fundamental Subspaces of $C$
The plane defined by the column space of \((\partial f/\partial x)\), however, does change orientation with the state vector evolution. This line and plane together span \(R^3\) under non-degenerate conditions. The orientation of the plane with respect to the line control the linear independence of the columns of the \(P_y\) matrix as described by equation (3.25). This condition is shown in figure (3.2).

As long as this line is not coplanar with the column space of \((\partial f/\partial x)\), \(P_y\) remains rank 3. If this line does fall in the plane defined by the column space, then the union of these two subspaces would not span \(R^3\) and consequently \(P_y\) would be singular. If the line lies nearly in the plane, then the \(P_y\) matrix is ill-conditioned. In this case, there will exist one direction in the \(y_1 - y_2 - y_3\) space, namely perpendicular to the column space of \((\partial f/\partial x)\), in which the first step state estimate is predicted to have very little uncertainty.

3.2 Numerical Example

A simple two state example problem with nonlinear dynamics and a nonlinear measurement equation is used to illustrate this anomaly in the two step filter. The geometry of this problem is shown in Figure 3.4. The kinematics consist of a particle following a spiral path defined by a constant angular velocity \(\omega_0\) and a constant radial velocity \(v_0\) as if the particle was attached to a string of increasing length. The nonlinear differential equations in Cartesian coordinates are

\[
\frac{dx_1}{dt} = \frac{x_1v_0}{\sqrt{x_1^2 + x_2^2}} - x_2\omega_0 \tag{3.26}
\]

\[
\frac{dx_2}{dt} = \frac{x_2v_0}{\sqrt{x_1^2 + x_2^2}} + x_1\omega_0 \tag{3.27}
\]

The linearized state dynamics matrix, \(A(\bar{x})\), used to propagate the state transition matrix by numerically integrating \(\Phi = A(\bar{x})\Phi\) is
Figure 3.4: Example Problem Geometry

\[
A = \begin{bmatrix}
\frac{x_2^2 v_0}{(x_1^2 + x_2^2)^{3/2}} & -\frac{\omega_0 x_2}{(x_1^2 + x_2^2)^{3/2}} - \omega_0 \\
-\frac{\omega_0 x_1 x_2}{(x_1^2 + x_2^2)^{3/2}} + \omega_0 & \frac{x_1^2 v_0}{(x_1^2 + x_2^2)^{3/2}}
\end{bmatrix}
\]  \hspace{1cm} (3.28)

The measurement is the range between the particle and a fixed point located at the coordinates (1, 0). This gives the measurement equation.

\[
\tilde{z} = \sqrt{(x_1 - 1)^2 + x_2^2}
\]  \hspace{1cm} (3.29)

The first step states are defined as the second step states augmented by the measurement, following the suggestion in [Haupt. et al. 1996].

\[
\tilde{y} = \begin{bmatrix}
\sqrt{(x_1 - 1)^2 + x_2^2} \\
x_1 \\
x_2
\end{bmatrix}
\]  \hspace{1cm} (3.30)

Hence \( H = [1 \quad 0 \quad 0] \) and the partial matrix, \( \frac{\partial f}{\partial x} \), is given by:

\[
\frac{\partial f}{\partial x} = \begin{bmatrix}
\frac{x_1 - 1}{\sqrt{(x_1 - 1)^2 + x_2^2}} & \frac{x_2}{\sqrt{(x_1 - 1)^2 + x_2^2}} \\
1 & 0 \\
0 & 1
\end{bmatrix}
\]  \hspace{1cm} (3.31)

Specific numbers used in the example and the filter implementations are all listed in table 3.1. The initial conditions are in error from the reference starting
Table 3.1: Numerical Data for Example Problem

<table>
<thead>
<tr>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Filtered Points</td>
<td>300001</td>
</tr>
<tr>
<td>Independent Variable (t)</td>
<td>0 : 6</td>
</tr>
<tr>
<td>Filter Initial State ($\tilde{x}_0$)</td>
<td>{2.0, 0.0}</td>
</tr>
<tr>
<td>True Initial State ($\mathcal{X}(0)$)</td>
<td>{2.583, 0.313}</td>
</tr>
<tr>
<td>Measurement Covariance (R)</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>Filter Initial Second Step State Covariance ($P_{x0}$)</td>
<td>$\text{diag}{0.25, 0.25}$</td>
</tr>
<tr>
<td>Filter Initial First Step State Covariance ($P_{y0}$)</td>
<td>$\begin{bmatrix} 0.226 &amp; 0.209 &amp; 0.000 \ 0.209 &amp; 0.247 &amp; 0.000 \ 0.000 &amp; 0.000 &amp; 0.248 \end{bmatrix}$</td>
</tr>
<tr>
<td>Second Step State Discrete Time Process Noise ($Q_d$)</td>
<td>$\text{diag}{10^{-12}, 10^{-12}}$</td>
</tr>
</tbody>
</table>

conditions (2, 0) and a normally distributed error is added to the measurements. A process noise term is included in the filter to prevent the filter from losing sensitivity to new measurements as $t \to \infty$ and $P_y, P_x \to 0$. No dynamic noise is simulated in the truth model, however. A plot of the particle motion and one set of noisy measurements is shown in Figure 3.5. A large number (30000) of data points were generated in order to illustrate the sharp decreases in eigenvalues of $P_y$ which can occur over a very small time period.

Eigenvalues of the $P_y(\cdot)$ matrix on the order of $10^{-15}$ occurred near $t \approx 1.3$ and $t \approx 4.8$. The matrix $C$ is computed from each point in the filter time history from the a posteriori state covariances and estimated state. The largest eigenvector of $C$ was used as the basis of the column space of $C$. Figure 3.6 plots the numerical values demonstrating the properties of $C$. The three singular values of $C$ are plotted in Figure 3.6(a). Note that the first singular value is on the order of $10^{18}$ larger than the other two, and consequently the rank of $C$ is 1 (property 1). The dot product between the "steady state" $\tilde{z}$, computed as the average of the last 5000 points, and each previous $\tilde{z}_i$ is plotted in Figure 3.6(b). This dot product is very nearly unity once the large initial state error has been reduced by the filter. Hence,
Figure 3.5: Example Problem State and Observations
it has been demonstrated that the column space of \( C \) is fixed (property 2).

The explanation for the constant basis of \( C \)'s column space is illustrated in Figure 3.6(c). In this figure the dot product between the \( H \) matrix and the basis vector \( \bar{e} \) is plotted. After \( t \approx 0.5 \) this product becomes very small in comparison to the magnitude of the \( H^T \) and \( \bar{e} \) vectors (both unity) indicating that the two vectors are orthogonal and therefore the column vector \( H^T \) is in the nullspace of \( C \). The small jump near \( t \approx 1.3 \) in Figure 3.6(c) is possibly caused by numerical problems resulting from the near singularity at that point. The magnitude of this deviation is still very small as compared to unity.

Figure 3.7 illustrates the rank test defined in equation (3.25) using a numerical tolerance of \( 10^{-4} \) on top of a plot of the eigenvalues of \( P_y(+) \). This test condition is computed from the steady state eigenvector of \( C \) and the partial derivative matrix computed along the true trajectory. The rank test correctly predicts the two locations of low eigenvalues in the \( P_y \) time history as shown on Figure 3.7. The statement made in section 3.1.2 that the property of the numerical rank of \( C \) can still be expected to hold true even when the numerical rank of \( P_y \) has dropped is illustrated by comparing figure 3.6 with 3.7. In figure 3.6, the smallest singular values are smaller than \( 10^{-18} \) times the magnitude of the next larger one. However, the tolerance on numerical rank used in figure 3.7 to identify when \( P_y \) drops rank is significantly larger, \( 10^{-4} \). Therefore, the properties of \( C \) can be expected to still remain numerically valid even in regions approaching the point where \( P_y \) has become ill-conditioned. This, of course, may not be valid if the filter diverges as a result of the ill-conditioned covariance matrix.

Next, it is demonstrated that the existence of low eigenvalues of \( P_y \) at certain points actually does represent situations in which some linear combination of elements of the first step state vector is expected to have very little error. This effect, however, does not result in the existence of a linear combination of
Figure 3.6. Properties of the Column Space of $C$ a.) $\text{rank}(C) = 1$, b.) Column Space of $C$ Approaches a Fixed Orientation in $\mathbb{R}^n$, c.) $H^T$ is in the Nullspace of $C$. 
Figure 3.7: Numerical Rank Test for Ill-Conditioned $P_y$
corresponding second step state estimates which also have very little error. One difficulty in answering these questions is that the covariance matrix is the mean square error over an ensemble of states randomly disturbed about the mean state. As pointed out earlier, the experience with Monte Carlo simulations indicates that the location of the low eigenvalues of $P_y$ depends upon the specific set of observations in addition to the initial conditions. At first one might assume that this suggests the low eigenvalues are not representative of a “real” condition of the mean square first step state error. One must remember, however, that for two-step filter, much like the EKF and I EkF, the covariance propagation is not independent of the state propagation. Hence an ensemble of filter simulations, even if started from the same initial conditions, do not represent the same time history of the evolution of $P_x$, $P_y$ and $\frac{\partial f}{\partial x}$. The evolution of these matrices is very important in determining the path taken by the column space of the $C$ matrix before it converges to a fixed orientation.

A series of five realizations of the example problem are generated for 8500 points, each starting from exactly the same initial conditions as given in table 3.1. The random measurement error produces a slightly different state estimate time history, however. The minimum eigenvalue of $P_y(\epsilon)$ for each of these five simulations is shown in figure 3.8 (a). A different $C$ matrix column vector $\bar{C}$ is computed from the average of the last 5000 points of each run and used in the test given in equation (3.25). Partial derivatives are computed along the true trajectory and a tolerance of $\epsilon = 10^{-4}$ is used. The results of this test are plotted in figure 3.8 (b). What this shows is that the wide difference in location of the points with low eigenvalues is the result of the different orientation of the column space of $C$ depending upon the state estimate time history. Recall that the equations (3.3) and (3.4), which are necessary for the $C$ matrix to remain fixed, are only approximations which are valid when equation (3.6) holds. This is not true at the
start of each filter simulation but rapidly becomes so, as the plots in figure 3.6 show. The different measurement and state estimate time histories for the five simulations in figure 3.8, owing to the random measurement noise, cause the $C$ matrix to take a different "path" to this converged orientation for each run. A different column space basis vector $\mathbf{c}$ is thus generated for each simulation.

This point is further emphasized by the plot in figure 3.8 (c). For this figure, the filter was run five times with exactly the same initial conditions but with perfect measurements for the first 1001 points. This forces the $C$ matrix to converge to the same orientation in each of the simulations before noisy data is encountered. On this plot two curves are shown; the minimum eigenvalue of the average of predicted $P_y(\cdot)$ matrices from the five runs; and the minimum eigenvalue of the average squared error $(\bar{y} - \hat{y})(\bar{y} - \hat{y})^T$. Both of these plots go through a minimum at the same point. This indicates that the anomaly discovered for the two-step estimator actually does predict a location where some linear combination of the first step state mean squared error reaches very low values.

This anomaly, however, does not effect the second step state estimate (assuming, of course, that it does not cause the iterative solution to fail), as figure 3.8(d) shows. In this figure, the two eigenvalues of the mean predicted $P_x(\cdot)$ are plotted along with the eigenvalues of the mean squared second step state error $(\bar{x} - \hat{x})(\bar{x} - \hat{x})^T$. The second step state error eigenvalues do not exhibit a similar drop at that location, so it can be concluded that this anomaly only has an effect on the first step states.

3.3 Consideration of Special Cases

The special conditions for the $C$ matrix column space enumerated earlier will be demonstrated for this simple case by using different measurement equations and first step state definitions. The three special cases considered will be: $l = m$. 

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Figure 3.8. Comparison of Minimum $eig(P_y)$ Location with Mean Squared First and Second Step State Error
$l > m$ and $n = m$. In the first two cases the first step state vector will be formed by augmenting the state vector by the measurement equation $f(x) = [h(x) \ f]^T$. For the $m = n$ case the first step state vector will be the measurement equation $f(x) = h(x)$. These different cases will be set up by adding range measurements from additional fixed points. The geometry of these three cases is illustrated in figure 3.9. Specific definitions of the first step states and measurement equations are given in table 3.2. For all cases the second step states are the same as in the previous section.

### 3.3.1 Special Case: $l = m$

In the case of $l = m$, the number of observations available at each update is equal to the number of second step states. As was shown above, the null space of the $C$ matrix approaches the column space of $H^T$. Because these two subspaces are of equal dimension the null space of $C$ is completely determined by the column space of $H^T$. Unlike in the general case, there are no free directions in this subspace. Figure 3.10 illustrates this. The rank of $C$ is shown to be 2 in part (a) of this figure, because $n - m = 2$. In part (b) it is demonstrated that the column
Table 3.2: Example Problems to Illustrate Special Cases

<table>
<thead>
<tr>
<th>Special Case</th>
<th>Measurement equation</th>
<th>First Step State Defn.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: (l = m)</td>
<td>(h(\bar{x}) = \left[ \frac{\sqrt{(x_1 - 1)^2 + x_2^2}}{\sqrt{(x_1 + 1)^2 + x_2^2}} \right])</td>
<td>(f(\bar{x}) = \left[ \frac{h(\bar{x})}{l} \right])</td>
</tr>
<tr>
<td>B: (l &gt; m)</td>
<td>(h(\bar{x}) = \left[ \frac{\sqrt{(x_1 - 1)^2 + x_2^2}}{\sqrt{(x_1 + 1)^2 + x_2^2}} \right] \frac{\sqrt{x_1^2 + (x_2 - 1)^2}}{\sqrt{x_1^2 + (x_2 - 1)^2}})</td>
<td>(f(\bar{x}) = \left[ \frac{h(\bar{x})}{l} \right])</td>
</tr>
<tr>
<td>C: (n = m)</td>
<td>(h(\bar{x}) = \left[ \frac{\sqrt{(x_1 - 1)^2 + x_2^2}}{\sqrt{(x_1 + 1)^2 + x_2^2}} \right])</td>
<td>(f(\bar{x}) = h(\bar{x}))</td>
</tr>
</tbody>
</table>
space of C rapidly converges to a fixed orientation. Part (c) of figure 3.10 shows that this space is orthogonal to the two column vectors of $H^T$ thereby demonstrating that the column space of $H^T$ completely defines the nullspace of C.

3.3.2 Special Case: $l > m$

For problems in which the dimension of the measurement vector exceeds the dimension of the second step state vector, it was shown that the column space of C does not remain in a fixed orientation. This is demonstrated in figure 3.11. Part (a) of this figure illustrates that the first property of the C matrix is unaffected by the condition of $l > m$ because the rank remains 1. Part (b) plots the components of a normalized eigenvector of the largest eigenvalue of C. This figure clearly indicates that the orientation of the column space of C does not converge to a fixed orientation in $R^n$ as more measurements are processed.

3.3.3 Special Case: $n = m$

In the case of the dimension of the first step state space and the second step state space being equal it was shown earlier that the two-step filter is similar to a coordinate transformation filter. To illustrate the behavior of the C matrix subspaces, this problem was run using the same two-step filter algorithm as the other cases (iterative second step solution). Figure 3.12 (a) plots the singular values of the matrix C showing that they are all vanishingly small, consistent with the results shown earlier. The filter with $n = m$ does, however, generate near singular $P_y$ covariance matrices as shown in figure 3.12(b). This is correlated exactly with the situations in which the partial derivative matrix $(\partial f/\partial x)$ becomes singular as shown in figure 3.12(c). It is easy to show that this occurs when $x_2 = 0$ which corresponds to a collinear arrangement of the particle and the ranging sources. The geometric interpretation of the situation is that small changes in position along the $x_1$ axis cause an equal change in both $y_1$ and $y_2$ whereas small
Figure 3.10. Special Case A: $l = m$. a.) Demonstration that $\text{rank}(C) = 2$. b.) Demonstration that the Column Space of $C$ is Fixed. c.) Demonstration that the Columns of $H^T$ are in the Nullspace of $C$. 
Figure 3.11. Special Case B: $l > m$, a.) Demonstration that $\text{rank}(C) = 1$, b.) Demonstration that the Column Space of $C$ Changes.
changes along the $x_2$ axis do not change the states $y_1$ and $y_2$ to first order.

3.4 Application to the Relative Navigation Filter

The analysis of the numerical rank-deficiency problem in the first step state covariance matrix developed in the first part of this chapter and then demonstrated on several simple example problems is now applied to the relative orbit determination problem. The same simulation results presented in figure (3.1) are used to compute the $C$ matrix; illustrate the properties of this matrix, and demonstrate the numerical rank test in equation (3.25).

Figure (3.13) shows the application of this theory to the orbital navigation problem. In the plot of figure (3.13)(a), the minimum eigenvalue of $P_x$ is given as a function of the true anomaly of the primary. This is a magnification of the lower curve on figure (3.1)(a). In addition to the point near $f_P \approx 1.1$ where the filter began to diverge, there is an earlier point just before $f_P \approx 0.5$ where the eigenvalue gets small, but stays positive. This point does not have an effect upon the filter operation because the covariance matrix is still large at that time. Figure (3.13)(b) plots the elements of the one dimensional column space of $C$ for this problem. Note that it converges to a fixed orientation by about $f_P \approx 0.5$. The rank test using equation (3.25) is applied to the state history of the filter with a numerical tolerance of $10^{-5}$ using the $\tilde{c}$ vector elements generated at $f_P = 0.714$ as $c_\infty$. The test identifies the location of the failure given only the present state estimate. This plot also illustrates that it can identify the earlier instance of low eigenvalues at $f_P \approx 0.5$.

3.5 Recommended Filter Modifications

Having shown that the occurrence of low eigenvalues in the first step covariance is a fundamental property of the two-step estimator, modifications to the filter are now suggested which reduce the effect of these anomalies on the filter.
Figure 3.12. Special Case C: $n = m$. a.) Demonstration that $\text{rank}(C) = 0$, b.) Illustration of a Small Eigenvalue of $P_y$, c.) Numerical Rank of the Partial Derivative Matrix.
Figure 3.13. Demonstration of the Identification of Locations of Low \( \text{eig}(P_y) \) On the Relative Navigation Problem. a.) Smallest Eigenvalue of \( P_y \), b.) Elements of the Basis Vector of \( C' \)'s Column Space, c.) Demonstration of the Rank Test Given in equation (3.25).
operation and reduces the likelihood that a non positive definite $P_y$ would occur and result in subsequent failure of the second step minimization. In order for these changes to be practical, their implementation should not degrade the filter performance such that it loses its advantage over the IEKF and other conventional nonlinear estimators. Use of a factored form, as mentioned at the start of this chapter, is suggested for all two-step filters. While UD covariance factorization does not eliminate negative eigenvalues or make the $P_y$ better conditioned, it allows the inverse of $P_y$ to be formed with more precision when it gets close to singular. Details of this algorithm are given in Appendix A.

3.5.1 Skipping Points in Neighborhood of Low $\text{eig}(P_y)$

The simplest modification to the two-step filter is to monitor some indication of the numerical rank of $P_y$ and eliminate processing of measurements and the prediction of states through arcs in which this matrix is numerically rank deficient. The test given in (3.25) can be used for this purpose although other indicators, such as the relative magnitudes of the largest and smallest diagonals of $D$, may be more practical for real time implementation. If $k$ points are removed from the set of observations, then the first step state and covariance propagations are

$$\tilde{y}_{i+k}(-) = \tilde{y}_i(+) + f(\tilde{x}_{i+k}(-)) - f(\tilde{x}_i(+)$$  \hspace{1cm} (3.32)

and

$$P_{y_{i+k}}(-) = P_{y_i}(+) + \frac{\partial f}{\partial x}\bigg|_{\tilde{x}_{i+k}(-)} P_{x_{i+k}}(-) \frac{\partial f}{\partial x}\bigg|_{\tilde{x}_{i+k}(-)} - \frac{\partial f}{\partial x}\bigg|_{\tilde{x}_i(+) P_{x_i}(+) \frac{\partial f}{\partial x}\bigg|_{\tilde{x}_i(+)}$$  \hspace{1cm} (3.33)

If these points are chosen such that the rank of equation (3.25) drops below $n$ at a point inside the interval $i$ to $i+k$, then the filter will not operate on those points and no ill-conditioned covariance matrices will be used in the second step iterations. Figure (3.14) illustrates the application of this method to the same
orbital problem shown earlier. In this case 20 points are manually deleted from the
observations near $f_T \approx 1.1$. This figure clearly shows that once the filter is
prevented from operation within a small interval near the singularity, it does not
diverge and continues to operate properly on the data following the potential
near-singularity.

This method has the disadvantage that some amount of data is discarded.
Furthermore, on-line application of this method requires the definition of a test
which is applied only to the present estimate of the states and their covariances.
Experience has shown that it is sometimes difficult in practical applications to
define such a test and set the threshold value for rejecting points. In chapter 6 such
a test, involving the numerical rank of $\partial f/\partial x$ for a filter with $n = m$ is used along
with the requirement that a minimum number of additional points are rejected
following the decision to reject a point. This procedure does work, and was found
to be more reliable than the next suggested modification. This technique has the
disadvantage that some number of otherwise "good" measurements are discarded.

It may also be practical to switch from the two-step filter to some other
estimator, such as the IEKF, through the arcs of deleted points in order to make
use of the otherwise discarded data. This would pose difficulties in re-initializing
the two-step filter once the deleted points have been processed by the other filter.
This is because operation of the other filter results in a better state estimate and a
different state covariance matrix than that from state propagation alone.
Furthermore, this new state and covariance would change the subsequent operation
of the two-step filter and move the location of the next near-singular $P_y$. It is
unclear as to what test would be applied to the updated second step state and
covariance generated by the other filter to determine when to switch back to the
original two-step filter.
Figure 3.14. The Effect of Manually Deleting Points in the Neighborhood of the Ill-Conditioned $P_y$
3.5.2 Addition of a Small Matrix to $P_y$ as "Process Noise"

Another simple modification to the existing two-step algorithm is the addition of a diagonal matrix of small positive terms to equation (2.53) so as to ensure that the resulting $P_y$ is always positive definite.

$$
P_{y_{i+1}}(-) = P_{y_i}(+) + \left. \frac{\partial f}{\partial x} \right|_{\dot{x}_{i+1}(-)} P_{\dot{x}_{i+1}}(-) \left. \frac{\partial f}{\partial x} \right|_{\dot{x}_{i+1}(+)}^T - \left. \frac{\partial f}{\partial x} \right|_{\dot{x}_i(+)} P_{\dot{x}_i}(+) \left. \frac{\partial f}{\partial x} \right|_{\dot{x}_i(+)}^T + \epsilon I
$$

(3.34)

This additional term enters the first step state covariance propagation very similar to "process noise". The value of the number $\epsilon$ is adjusted manually through simulations so as to make the covariance time history as close to continuous across a near-singularity as possible. The presence of this term prevents the covariance bounds from decreasing below some value as the filter reaches steady state. This has the same effect as the process noise term in a conventional Kalman filter.

It has already been noted that the presence of process noise in the second step state propagation will not reduce the possibility of numerically small eigenvalues of $P_y$. The definition of $C$ makes no assumption about how $P_x$ is propagated between measurements and the expression of the filter in terms of the $C$ matrix does not include $P_x$ as a separate term. Hence all of the properties derived earlier would be exactly the same regardless of the propagation of the second step state covariance.

The application of this method to the previous example is shown in figure 3.15 in which the same case is run, but a term is added as in equation (3.34). A numerical value of $\epsilon = 10^{-20}$ is used. This improvement is shown on that figure where the eigenvalues do not extend below the order of $10^{-20}$, always remaining positive. The state estimates, as shown in figure (3.15) do not diverge and the error stays within the covariance bounds for the full time history. Selection of the value $\epsilon$ is done manually by comparing the results of simulations. The sensitivity of the filter to this value is shown in figure 3.15 for the same problem in figure 3.13. A
value of $\epsilon = 10^{-25}$ is too small, and does not eliminate the failure. A value of $\epsilon = 10^{-20}$ reaches prevents the ill-conditioned $P_y$ while having minimal effect upon the filter operation otherwise.

One point which should be noted is that the properties derived earlier for the $C$ matrix do not apply to the filter modified by the $\epsilon I$ term in equation (3.34). The experience, however, is that the basis of the $C$ column space will remain constant during the period of the state time history before the first near singularity is encountered. From that point on, however, this property is not found to always be valid. It is thought that this is because at the points near where one eigenvalue reaches its minimum, this small additional matrix term has the largest effect upon the covariance propagation. Remember that this additional term is an artificial modification to the filter to prevent a numerical problem, that of the difficulty in using an ill-conditioned $P_y$, which results from a real property of the two-step filter, that of occasionally generating second step state estimates which have very little mean square error in some linear combinations. This modification is intended to be applied in such a way as to eliminate the numerical difficulties while having a minimal effect upon the operation of the filter otherwise.

### 3.6 Change of Variables

Conceptually, if the function used to define the first step states is changed at some time preceding the predicted occurrence of ill-conditioned $P_y$, it may be possible to choose a set of states which will have a longer time before the next occurrence of low $\text{eig}(P_y)$. This procedure could be repeated again as the newer, but more distant, anomaly is approached. Although a theoretical investigation into this modification of the filter is beyond the scope of this study, it is mentioned here as a conceptual solution to this problem. It may also be possible, in some special conditions, to choose a set of first step states such that an ill-conditioned $P_y$ is
Figure 3.15. Sensitivity of the Relative Navigation Two-Step Filter to Magnitude of \( \epsilon \). a.) \( \epsilon = 10^{-25} \), b.) \( \epsilon = 10^{-20} \).
never encountered within the expected range of filter operation.

The change of variables that this requires must be a nonlinear function. A linear coordinate transformation of the first step states does not change the existence and location of the low eigenvalues. This is shown by defining the transformed first step state vector as \( \bar{y} = Ty \). The corresponding partial derivative matrix and first step covariance are similarly transformed according to

\[
P'_y = TP_yT^T \quad \text{and} \quad \partial f' / \partial x = T \partial f / \partial x.
\]

Substituting these into the \( C \) matrix definition for the transformed states results in \( TCT^T \) which has a column space basis of \( \{ T \bar{c}_1, ...T \bar{c}_2, ...T \bar{c}_{n-m} \} \). In the transformed coordinates, the rank test, given by equation (3.25) is expressed as

\[
T \begin{bmatrix}
\frac{\partial f}{\partial \bar{x}}
\bar{c}_1, \bar{c}_2, ..., \bar{c}_{n-m}
\end{bmatrix}
\]  

(3.35)

This rank test has have the same rank as the original test, hence the low eigenvalues of \( P_y \) will occur at the same points as in the original definition of \( \bar{y} \).
CHAPTER 4

SYSTEM MODELS

This chapter defines the measurements and dynamics in the "truth" model used to generate simulated data for validating the navigation filters. It also includes a study of linearized models and the magnitude of the errors produced by them. Chapter 5 covers the actual mechanization of these models when used as part of the navigation filters.

4.1 Coordinate Systems

Three different coordinate systems are required to describe the true motion of the primary and secondary spacecraft in this study. These are shown in figure 4.1. An Earth-centered inertial (ECI) reference frame, indicated by capital letters, is defined with X and Y at some fixed orientation in the equatorial plane and Z perpendicular to the equatorial plane. Conventional practice in orbital mechanics is to defined X in the direction of the first point in Aries [Buglia, 1988]. In all of the filter simulations, the primary orbit is assumed known and is not maneuvering. Hence, for convenience in the simulations, the X axis is aligned with the line of nodes of the primary vehicle. This can also be expressed as \( \Omega_P = 0 \).

The orbit of the primary vehicle is specified by the Keplerian orbit elements: argument of perigee \( (\omega_P) \), inclination \( (i_P) \), eccentricity \( (e_P) \), semimajor axis \( (a_P) \) and true anomaly \( (f_P) \). The three angles, \( \omega_P, f_P, \) and (trivially) \( \Omega_P \), define the orientation of the primary orbit as shown on figure 4.1. The two elements \( a_P \) and \( e_P \) define the shape of the orbit and the vehicle motion with time.

The orbit local reference frame is a non-inertial reference with its origin at
Figure 4.1: Definition of the Coordinate Systems
the primary center of mass and $x$ along the radial direction from the center of the Earth, $z$ perpendicular to the orbit plane, and $y$ completing the right handed triad and oriented in the direction of orbital motion. The orbit elements of the primary vehicle are used to transform the relative position of the secondary vehicle between the inertial reference and the orbit local reference

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos(\omega_P + f_P) & \cos i_P \sin(\omega_P + f_P) & \sin i_P \sin(\omega_P + f_P) \\ -\sin(\omega_P + f_P) & \cos i_P \cos(\omega_P + f_P) & \sin i_P \cos(\omega_P + f_P) \\ 0 & -\sin i_P & \cos i_P \end{bmatrix} \begin{bmatrix} X_S - X_P \\ Y_S - Y_P \\ Z_S - Z_P \end{bmatrix}$$

(4.1)

A body fixed coordinate system, indicated by \{x', y', z'\} on figure 4.1, could be defined along any convenient reference attached to the primary spacecraft itself.

This reference frame is related to the orbit local reference by a transformation matrix which is a function of the primary vehicle attitude quaternion ($\tilde{q}_P$).

$$\tilde{z'} = T(\tilde{q}_P)\tilde{z}$$

(4.2)

This reference frame is, of course, non-inertial and the transformation given symbolically in equation (4.2) requires the output of an attitude determination system. The estimation of attitude was not considered as part of this study and is assumed to be known perfectly.

With these definitions, the positions and velocities of the primary and secondary at any time are expressed by the 12 quantities: $X_S - X_P$, $Y_S - Y_P$, $Z_S - Z_P$, $U_S - U_P$, $V_S - V_P$, $W_S - W_P$, $a_P$, $e_P$, $\omega_P$, $f_P$, $i_P$, $\Omega_P$. For the filter to be developed in the next chapter, five of the primary vehicle orbit elements are assumed to be known perfectly and remain constant. This is because motion of the primary is assumed to be described by classical two-body orbit dynamics. True anomaly of the primary ($f_P$) is used as the independent variable. The state to be estimated by the filter is therefore six elements, representing the differences in position and velocity between the primary and the secondary.
Table 4.1: Intersatellite Radar Performance

<table>
<thead>
<tr>
<th>Operating Range Max. (m)</th>
<th>Geostationary Transfer</th>
<th>Apollo-Soyuz</th>
<th>Space Shuttle</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>&gt; 100 km</td>
<td>&gt; 100 km</td>
<td>550 km</td>
</tr>
<tr>
<td>Range Error (1σ, m)</td>
<td>100 m @ 100 km</td>
<td>0.003 ρ</td>
<td>0.003 ρ</td>
</tr>
<tr>
<td></td>
<td>4.7 m @ 5 km</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>4.5 m or 1.6 m @ 100 m</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Range Rate Error (1σ, m/sec)</td>
<td>0.5 m/s @ 100 km</td>
<td>0.1 m/s</td>
<td>0.1 m/s</td>
</tr>
<tr>
<td></td>
<td>0.05 m/s @ &lt; 5km</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Angle Error (1σ, radian)</td>
<td>8.7 × 10⁻³ @ 100 km</td>
<td>6.7 × 10⁻¹</td>
<td>2.7 × 10⁻³</td>
</tr>
<tr>
<td></td>
<td>1.7 × 10⁻³ @ 20 m - 5 km</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

†dependent upon clock frequency.

4.2 Measurement Model

At the intersatellite distances considered in this study (up to hundreds of kilometers), a radio-frequency (RF) two way radar system is selected as the best ranging sensor to determine the six relative states defined earlier. That choice is also motivated by the desire for the minimum number of interfaces required on the secondary spacecraft. For such large distances, however, some transponder is required on the secondary to relay the radar pulse back to the primary. It is assumed that the primary has its own orbit determination system in addition to the relative navigation system described in this study. Expected accuracy of radar systems operating over those distances is inferred from past spaceflight experience and proposed mission designs [Erwin and Krishen, 1985], [ESA, 1983]. A summary of this past experience is given in table 4.1.

With these data available, a model for the radar sensor is assumed in which the range error and range rate error are zero-mean, uncorrelated, white. Gaussian random processes having standard deviations of 0.3 % of the range and 0.1 m/sec respectively. The angle measurements, giving the relative elevation and azimuth
between the primary and the secondary, are also assumed uncorrelated Gaussian white noise processes with standard deviations of 0.0027 rad.

The noisy measurements of range and range rate are computed from the true position and velocity as follows:

\[
\rho = \sqrt{(X_S - X_P)^2 + (Y_S - Y_P)^2 + (Z_S - Z_P)^2 + \eta} \\
\dot{\rho} = \frac{(X_S - X_P)(U_S - U_P) + (Y_S - Y_P)(V_S - V_P) + (Z_S - Z_P)(W_S - W_P)}{\sqrt{(X_S - X_P)^2 + (Y_S - Y_P)^2 + (Z_S - Z_P)^2}} + \zeta
\]

where \(\eta\) and \(\zeta\) are white Gaussian noise processes with mean and covariances

\[
\langle \eta \rangle = 0 \\
\langle \eta^2 \rangle = 9 \times 10^{-6} \rho^2 \\
\langle \zeta \rangle = 0 \\
\langle \zeta^2 \rangle = 0.01 \text{ m}^2/\text{sec}^2 \\
\langle \eta \zeta \rangle = 0
\]

In order to define angle measurements a reference frame fixed on the body of the secondary spacecraft at some known orientation with respect to the receiving antennas on that vehicle must be specified. As described earlier, the transformation between this reference frame and the local reference frame is a function of the primary vehicle attitude. In this study, the attitude determination problem was not considered. Angles are therefore defined with respect to the local coordinate system, recognizing that this implies perfect knowledge of the secondary vehicle attitude.

The geometry defining the elevation and azimuth measurements is shown in figure 4.2. The measurement equations for the direction cosines given the primary vehicle attitude quaternion, \(\hat{q}_P\), are

\[
\begin{bmatrix}
\cos \alpha \\
\cos \beta
\end{bmatrix} = T(\hat{q}_P) \begin{bmatrix}
\frac{\hat{z}}{\rho} \\
\frac{\hat{z}}{\rho}
\end{bmatrix} + \begin{bmatrix}
\nu \\
\xi
\end{bmatrix}
\]

\[
(4.6)
\]
For this study $T = I$ is assumed and the direction cosines themselves are used as the measurements. The transformation between the inertial coordinates and the orbit local coordinates as described in the previous section is used to get $x$ and $z$. The errors are again assumed to be white Gaussian random variables with mean and covariance given by:

\[
\begin{align*}
\langle \nu \rangle &= 0 \\
\langle \nu^2 \rangle &= 7.29 \times 10^{-6} \text{ rad}^2 \\
\langle \xi \rangle &= 0 \\
\langle \xi^2 \rangle &= 7.29 \times 10^{-6} \text{ rad}^2 \\
\langle \nu \xi \rangle &= 0
\end{align*}
\]

A random number generator provided as part of commercially available simulation software [The Mathworks, Inc. 1992] which makes use of the algorithm in [Forsythe et al., 1977] is used to generate simulated measurement noise for the observations. An important consideration in using random number generators is that the numbers which they produce do not introduce any biases over typical
lengths of sequences required in the simulation. The random number generator is expected to generate samples from the normal distribution, $N(0, 1)$. To validate the random number generator, a series of $2 \times 10^7$ samples are generated and the running mean and mean square value of that sequence are computed. These are shown in figure 4.3. The first plot in this figure shows the running mean of the sequence plotted against the number of samples used to compute the mean. The bounds for a 95% confidence interval [Papoulis 1982], [Devore 1982]

$$-1.967 \frac{\sigma}{\sqrt{N}} < \langle x \rangle < 1.967 \frac{\sigma}{\sqrt{N}}$$  (4.8)

are shown as solid lines on that figure. Note that all of the points remain inside the bounds. Therefore, it is assumed that the sequence may be treated as unbiased.

The lower plot in figure (4.3) is the running mean squared value. The 95% confidence interval for the covariance is

$$\frac{N \sigma^2}{\chi^2_{1-\delta/2}(N)} < \langle x^2 \rangle < \frac{N \sigma^2}{\chi^2_{\delta/2}(N)}$$  (4.9)

in which $\chi^2(N)$ is the Chi-squared percentile of $N$ degrees of freedom. For a large sample size, this can be approximated as

$$\chi^2_{1-\delta/2}(N) \approx \frac{1}{2} \left( 1.967 + \sqrt{2N - 1} \right)^2$$  (4.10)

and

$$\chi^2_{\delta/2}(N) \approx \frac{1}{2} \left( -1.967 + \sqrt{2N - 1} \right)^2$$  (4.11)

for a 95% confidence interval. Again, note that all of the points stay within these bounds. This shows that it is expected that the variance of this sequence is unity.

4.3 Orbital Dynamics

The orbits used to generate "truth" for the filter validation and Monte Carlo simulations are obtained from basic two-body orbital dynamics. For the
Figure 4.3: Validation of the Random Number Generator
two-body truth model, the primary and secondary are independently propagated in
the Earth-centered inertial system defined above.

\[
\begin{align*}
\frac{dX_p}{dt} &= U_S, \quad \frac{dY_p}{dt} = U_P \\
\frac{dY_p}{dt} &= V_S, \quad \frac{dY_p}{dt} = V_P \\
\frac{dZ_p}{dt} &= W_S, \quad \frac{dZ_p}{dt} = W_P \\
\frac{dX_p}{dt} &= -\frac{\mu X_p}{R_p^3}, \quad \frac{dW_p}{dt} = -\frac{\mu X_p}{R_p^3} \\
\frac{dY_p}{dt} &= -\frac{\mu Y_p}{R_p^3}, \quad \frac{dW_p}{dt} = -\frac{\mu Y_p}{R_p^3} \\
\frac{dZ_p}{dt} &= -\frac{\mu Z_p}{R_p^3}, \quad \frac{dW_p}{dt} = -\frac{\mu Z_p}{R_p^3}
\end{align*}
\]  \hspace{1cm} (4.12)

In (4.12) \( \mu \) is the gravitational parameter and a numerical value of
\( \mu = 3.986005 \times 10^{-14} \text{m}^3/\text{sec}^2 \) is used for it in all of the simulations. The radius
from the center of the Earth to the primary and secondary vehicles are given by
\( R_P \) and \( R_S \), respectively.

The differences between the primary and secondary position and velocity in
inertial space are taken at 1 second intervals and are then used to compute a
simulated measurement set. Measurement noise is added to these observations
using the random number generator. This measurement set, along with the true
anomaly of the primary \( f_P \), and the set of four constant primary vehicle orbit
elements \( a_P, e_P, i_P, \omega_P \) are sent to the filter as observations.

4.4 Linearized Orbital Dynamics

A closed form state transition matrix approximating the relative motion of
two satellites in local coordinates is now developed [Garrison, et al., 1995]. This is
used in a preliminary study of the effects of nonlinearities on the relative dynamics
in an elliptical orbit and as a comparison to existing linearized approximations to
the relative motion problem. The derivation of the state transition matrix begins
by expressing the inertial Cartesian coordinates of one vehicle in terms of a set of
Keplerian orbit elements. Orbit elements were chosen such that, in the absence of
disturbances or control, five will remain constant and the true anomaly will be time varying. The expressions for the positions and velocities are then linearized with respect to the primary (reference) orbit and transformed into the local coordinate frame attached to the primary vehicle.

These steps define a 6 by 6 matrix, \( \Psi(f_p, \alpha_p) \), which relates differences in the Keplerian elements \( (\delta \alpha) \) of the two orbits to small position and velocity differences between the primary and secondary. Starting with orbit elements, which already are integrals of two-body motion, allows one to immediately begin with a state transition matrix formulation as opposed to linearizing a differential equation formulation and then solving the resultant linear set as was done by others [Tschauner and Hempel, 1965], [Shulman and Scott, 1966]. The simple definition of the \( \Psi \) matrix provides a straightforward mechanism for determining values of the secondary position and velocity states in the primary frame as a function of the small differences in a set of orbit elements.

\[
\mathcal{E}(f_p) = \Psi(f_p, \alpha_p) \delta \alpha(f_p)
\]  

The \( \Psi(f_p, \alpha_p) \) matrix is conveniently partitioned into in-plane (local \( x \) and \( y \)) and out-of-plane (\( z \)) components and is only a function of the constant orbit elements of the primary vehicle (\( \alpha_p \)) and the true anomaly (\( f_p \)) of the primary vehicle. The true anomaly of the primary vehicle is used as the independent variable. This eliminates the requirement to numerically solve Kepler's equation at each time step. This approximation to Kepler's equation will, of course, have some inherent linearization error.

The next step is to analytically invert the \( \Psi \) matrix and obtain an expression which transforms from relative Cartesian positions and velocities in the local reference frame to Keplerian element perturbations, again as a function of the primary true anomaly. This second transformation is computed at a reference
epoch, \( f_P \).

\[
\delta \tilde{a}(f_P) = \Psi^{-1}(f_P, \alpha_P) \tilde{\epsilon}(f_P)
\]  

(4.14)

A separate linear mapping for the propagation of the orbit element difference between the primary and secondary orbits with respect to the primary orbit is derived. This is expressed as a state transition matrix which is a function of the starting and ending times. These times are expressed as the primary vehicle true anomalies, \( f_{P_1} \) and \( f_{P_2} \).

\[
\delta \tilde{a}(f_{P_1}) = \Sigma(f_{P_2}, f_{P_1}, \alpha_P) \delta \tilde{a}(f_{P_1})
\]  

(4.15)

Combining these two ideas; linear transformation from small differences in classical orbit elements to small local positions and velocities (the \( \Psi \) matrix) with linear propagation of small differences in classical orbit elements (the \( \Sigma \) matrix) produces a closed form state transition matrix giving the propagation of the local position and velocity.

\[
\tilde{\epsilon}(f_{P_1}) = \Psi(f_{P_2}, \alpha_P) \Sigma(f_{P_2}, f_{P_1}, \alpha_P) \Psi^{-1}(f_{P_1}, \alpha_P) \tilde{\epsilon}(f_{P_1})
\]  

(4.16)

This state transition matrix expresses the relationship between the measurable relative quantities (\( \tilde{\epsilon} \)) in terms of the parameters describing the primary orbit (\( \alpha_P \) and \( f_P \)).

The state transition matrix described above provides a straightforward means to map relative separations of the vehicles from one part of the reference orbit to any other part.

4.4.1 Transformation Matrix: \( \Psi(f_P, \alpha_P) \)

The derivation of the linearized dynamics begins with a basic expression for the position of the secondary spacecraft, in the ECI coordinate system, in terms of a set of Keplerian orbit elements [Battin, 1987]. These equations are expressed in
terms of the coordinates of the primary vehicle and small differences from them, all in inertial coordinates. Similarly, the secondary orbit element vector is expressed as a set of small differences from the elements of the primary orbit. This is expanded in terms of the small differences in the orbit elements and all higher order products of these are neglected.

These small relative disturbances in ECI coordinates are then rotated to the local coordinate system in figure 4.1. Performing this rotation gives the local coordinates of the secondary spacecraft relative to the primary as linear combinations of small orbit element differences.

\[
\frac{x}{r_p} = \frac{\delta a}{a_p} - \frac{2e_p + (e_p^2 + 1) \cos f_p}{(1 + e_p \cos f_p)(1 - e_p^2)} \delta e + \frac{e_p \sin f_p}{1 + e_p \cos f_p} \delta f \tag{4.17}
\]

\[
\frac{y}{r_p} = \delta f + \delta \omega + \cos i_P \delta \Omega \tag{4.18}
\]

\[
\frac{z}{r_p} = -\cos(\omega_P + f_P) \sin i_P \delta \Omega + \sin(\omega_P + f_P) \delta i \tag{4.19}
\]

As mentioned earlier, true anomaly of the primary is used as the independent variable throughout this analysis. With this in mind, equations (4.17), (4.18) and (4.19) are differentiated with respect to the primary true anomaly \( f_P \), the independent variable, to produce the three expressions for the relative velocity of the secondary in the local reference frame.

\[
\frac{u}{r_p} = \frac{1}{1 + e_p \cos f_p} \left[ -\frac{e_p \sin f_p}{2} \frac{\delta a}{a_p} + \sin f_p \frac{\delta e}{1 - e_p^2} + e_p \cos f_p \delta f \right] \tag{4.20}
\]

\[
\frac{v}{r_p} = -\frac{3\delta a}{a_p} + \frac{3e_p + (2 + e_p^2) \cos f_p}{1 + e_p \cos f_p} \frac{\delta e}{1 - e_p^2} - \frac{e_p \sin f_p}{1 + e_p \cos f_p} \delta f + \frac{e_p \sin f_p}{1 + e_p \cos f_p} \delta \omega + \frac{e_p \sin f_p \cos i_P}{1 + e_p \cos f_p} \delta \Omega \tag{4.21}
\]

\[
\frac{w}{r_p} = \frac{\sin i_P (\sin(\omega_P + f_P) + e_p \sin \omega_P)}{1 + e_p \cos f_p} \delta \Omega + \frac{\cos(\omega_P + f_P) + e_p \cos \omega_P}{1 + e_p \cos f_p} \delta i \tag{4.22}
\]

In these expressions, velocity terms are with respect to the true anomaly of the primary vehicle: \( u = dx/df_P \), \( v = dy/df_P \), \( w = dz/df_P \).
The local state vector is then conveniently nondimensionalized as the following:

\[ \bar{\mathbf{x}} \equiv \begin{bmatrix} \frac{(1+\varepsilon \cos f_p)z}{r_p} \\ \frac{\nu}{r_p} \\ \frac{(1+\varepsilon \cos f_p) \nu}{r_p} \\ \frac{(1+\varepsilon \cos f_p) \nu}{r_p} \\ \frac{\dot{\nu}}{r_p} \\ \frac{(1+\varepsilon \cos f_p) \omega}{r_p} \end{bmatrix} \]  \hspace{1cm} (4.23)

and small differences in orbit elements are linearly combined as the following:

\[ \delta \mathbf{\alpha} \equiv \begin{bmatrix} \frac{\delta a}{a_p} \\ \delta e \\ \delta f \\ \delta \omega + \cos i \delta \Omega \\ \sin i \delta \Omega \\ \delta i \end{bmatrix}. \hspace{1cm} (4.24) \]

This converts the system of (4.17) through (4.22) into the transformation matrix \( \Psi(f_p, \alpha P) \) defined in (4.13). This matrix is independent of the primary vehicle inclination \( (i_p) \) and in it the in-plane dynamics are decoupled from the out-of-plane dynamics. This matrix expresses the transformation from small differences in the orbit elements that are defined in (4.24), all of which are constant except for \( \delta f \), to the physically measurable local position and velocity. \( \Psi(f_p, \alpha P) \) is a function of the true anomaly of the primary orbit \( (f_p) \) and the vector of constant primary orbit elements \( (\bar{\alpha} P) \). The \( \alpha P \) vector has been reduced to only two elements \( \{(e_p, \omega_p)\} \) through the definition of \( \delta \alpha \) in equation (4.24) and the definition of the reference orbit node such that \( \Omega_p \equiv 0 \). A full listing of the elements of the \( \Psi \) matrix is given in table 4.2.
Table 4.2: Linearized Model Transformation Matrix: $\Psi$

\[
\begin{align*}
\Psi_{1,1} &= 1 + e_p \cos f_p & \Psi_{1,2} &= -\frac{2e_p+(e_p^2+1)\cos f_p}{1-e_p^2} \\
\Psi_{1,3} &= e_p \sin f_p & \Psi_{1,4} &= 0 \\
\Psi_{1,5} &= 0 & \Psi_{1,6} &= 0 \\
\Psi_{2,1} &= 0 & \Psi_{2,2} &= 0 \\
\Psi_{2,3} &= 1 & \Psi_{2,4} &= 1 \\
\Psi_{2,5} &= 0 & \Psi_{2,6} &= 0 \\
\Psi_{3,1} &= -\frac{e_p}{2} \sin f_p & \Psi_{3,2} &= \frac{\sin f_p}{1-e_p^2} \\
\Psi_{3,3} &= e_p \cos f_p & \Psi_{3,4} &= 0 \\
\Psi_{3,5} &= 0 & \Psi_{3,6} &= 0 \\
\Psi_{4,1} &= -\frac{3}{2}(1+e_p \cos f_p) & \Psi_{4,2} &= \frac{3e_p+(2+e_p^2)\cos f_p}{1-e_p^2} \\
\Psi_{4,3} &= -e_p \sin f_p & \Psi_{4,4} &= e_p \sin f_p \\
\Psi_{4,5} &= 0 & \Psi_{4,6} &= 0 \\
\Psi_{5,1} &= 0 & \Psi_{5,2} &= 0 \\
\Psi_{5,3} &= 0 & \Psi_{5,4} &= 0 \\
\Psi_{5,5} &= -\cos(\omega_p+f_p) & \Psi_{5,6} &= \sin(\omega_p+f_p) \\
\Psi_{6,1} &= 0 & \Psi_{6,2} &= 0 \\
\Psi_{6,3} &= 0 & \Psi_{6,4} &= 0 \\
\Psi_{6,5} &= \sin(\omega_p+f_p) + e_p \sin \omega_p & \Psi_{6,6} &= \cos(\omega_p+f_p) + e_p \cos \omega_p
\end{align*}
\]
Table 4.3: Linearized Model Inverse Transformation Matrix: $\Psi^{-1}$

\[
\begin{align*}
\Psi_{1,1}^{-1} &= \frac{2(3ep \cos fp + 2ep^2 \cos^2 fp)}{(1 - ep^2)(1 + ep \cos fp)} & \Psi_{1,2}^{-1} &= \frac{2ep(1 + ep \cos fp) \sin fp}{1 - ep^2} \\
\Psi_{1,3}^{-1} &= \frac{2ep \sin fp}{1 - ep^2} & \Psi_{1,4}^{-1} &= \frac{2(2ep \cos fp + 1 + ep^2 \cos^2 fp)}{(1 - ep^2)(1 + ep \cos fp)} \\
\Psi_{1,5}^{-1} &= 0 & \Psi_{1,6}^{-1} &= 0 \\
\Psi_{2,1}^{-1} &= \frac{3 \cos fp + 2ep + ep \cos^2 fp}{1 + ep \cos fp} & \Psi_{2,2}^{-1} &= \frac{2ep \cos fp + 2ep \cos^2 fp}{1 + ep \cos fp} \sin fp \\
\Psi_{2,3}^{-1} &= \frac{\sin fp}{1 + ep \cos fp} & \Psi_{2,4}^{-1} &= \frac{2 \cos fp + ep + ep \cos^2 fp}{1 + ep \cos fp} \\
\Psi_{2,5}^{-1} &= 0 & \Psi_{2,6}^{-1} &= 0 \\
\Psi_{3,1}^{-1} &= \frac{-\sin fp(3 + ep \cos fp)}{ep(1 + ep \cos fp)} & \Psi_{3,2}^{-1} &= \frac{(2 + ep \cos fp) \sin^2 fp}{1 + ep \cos fp} \\
\Psi_{3,3}^{-1} &= \frac{\cos fp}{ep} & \Psi_{3,4}^{-1} &= \frac{-2 + ep \cos fp}{ep(1 + ep \cos fp)} \sin fp \\
\Psi_{3,5}^{-1} &= 0 & \Psi_{3,6}^{-1} &= 0 \\
\Psi_{4,1}^{-1} &= \frac{\sin fp(3 + ep \cos fp)}{ep(1 + ep \cos fp)} & \Psi_{4,2}^{-1} &= \frac{-1 - ep \cos^2 fp - 2 \cos^2 fp}{1 + ep \cos fp} \\
\Psi_{4,3}^{-1} &= \frac{\cos fp}{ep} & \Psi_{4,4}^{-1} &= \frac{\sin fp(2 + ep \cos fp)}{ep(1 + ep \cos fp)} \\
\Psi_{4,5}^{-1} &= 0 & \Psi_{4,6}^{-1} &= 0 \\
\Psi_{5,1}^{-1} &= 0 & \Psi_{5,2}^{-1} &= 0 \\
\Psi_{5,3}^{-1} &= 0 & \Psi_{5,4}^{-1} &= 0 \\
\Psi_{5,5}^{-1} &= \frac{-\cos(\omega_p + fp) + ep \cos \omega_p}{1 + ep \cos fp} & \Psi_{5,6}^{-1} &= \frac{\sin(\omega_p + fp)}{1 + ep \cos fp} \\
\Psi_{6,1}^{-1} &= 0 & \Psi_{6,2}^{-1} &= 0 \\
\Psi_{6,3}^{-1} &= 0 & \Psi_{6,4}^{-1} &= 0 \\
\Psi_{6,5}^{-1} &= \frac{\sin(\omega_p + fp) + ep \sin \omega_p}{1 + ep \cos fp} & \Psi_{6,6}^{-1} &= \frac{\cos(\omega_p + fp)}{1 + ep \cos fp}
\end{align*}
\]
4.4.2 Orbit Element State Transition Matrix: $\Sigma(f_{P_2}, f_{P_1}, \alpha_P)$

The next step in the derivation is to approximate the relationship between the separation in true anomaly at one epoch to that at another. This, combined with equations (4.13) and (4.14) gives an expression for the evolution of the position of the secondary in the primary reference frame. Kepler’s equation is used at the two primary vehicle orbit positions given by the true anomalies $f_{P_1} \equiv f_P(t_1)$ and $f_{P_2} \equiv f_P(t_2)$ to derive a relationship between the eccentric anomaly of each vehicle at those two times.

$$E_2 - e_P \sin E_2 = E_1 - e_P \sin E_1 - \sqrt{\frac{\mu}{a_P^3}}(t_1 - t_2)$$  \hspace{1cm} (4.25)

The eccentric anomaly $(E_2, E_1)$ for a general secondary relative to the primary at two specific times is expressed as a displacement $(\delta E_2, \delta E_1)$ from the primary orbit $(E_{P_2}, E_{P_1})$ as

$$\delta E_2 = \left[ \frac{\sin f_{P_2}}{\sqrt{1 - e_P^2}} - \frac{1 + e_P \cos f_{P_2} \sin f_{P_1}}{1 + e_P \cos f_{P_1}} \right] \frac{\delta e}{1 + e_P \cos f_{P_1}} \delta E_1 +$$

$$\frac{3}{2} \left[ \frac{1 + e_P \cos f_{P_2}}{1 - e_P^2} (E_{P_2} - E_{P_1}) + e_P \left( \frac{\sin f_{P_2}}{\sqrt{1 - e_P^2}} - \frac{1 + e_P \cos f_{P_2} \sin f_{P_1}}{1 + e_P \cos f_{P_1}} \right) \right] \delta \alpha$$  \hspace{1cm} (4.26)

Again, all higher order products of components of $\delta \alpha$ have been neglected.

This is converted to true anomaly, by differentiating the equation

$$\tan \frac{f}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2}$$  \hspace{1cm} (4.27)

and expressing it in terms of $\delta f$ at the two orbital positions to produce an expression of the form

$$\delta f_{P_2} = A \frac{\delta \alpha}{a_P} + B \delta e + C \delta f_{P_1}$$  \hspace{1cm} (4.28)

in which $A$, $B$, and $C$ are defined as:

$$A = \frac{3}{2} \left[ \frac{(1 + e_P \cos f_{P_2})^2}{1 - e_P^2} \left[ \frac{E_{P_2} - E_{P_1}}{\sqrt{1 - e_P^2}} + e_P \left( \frac{\sin f_{P_2}}{1 + e_P \cos f_{P_2}} - \frac{\sin f_{P_1}}{1 + e_P \cos f_{P_1}} \right) \right] \right]$$  \hspace{1cm} (4.29)
\[
B = \frac{1}{1 - \varepsilon_P^2} \left[ (2 + \varepsilon_P \cos f_P) \sin f_P - \left( \frac{1 + \varepsilon_P \cos f_P}{1 + \varepsilon_P \cos f_P} \right)^2 (2 + \varepsilon_P \cos f_P) \sin f_P \right]
\]

(4.30)

\[
C = \left( \frac{1 + \varepsilon_P \cos f_P}{1 + \varepsilon_P \cos f_P} \right)^2
\]

(4.31)

This is the state transition matrix showing the evolution of the orbit element \((\delta \alpha)\) state vector with the motion of the primary spacecraft, as given in equation (4.15).

The \(\Sigma\) matrix is given below:

\[
\Sigma(f_P, f_{P1}, \alpha_{P1}) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
A & B & C & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

(4.32)

4.4.3 Local Coordinates State Transition Matrix: \(\Phi(f_{P2}, f_{P1}, \alpha_{P1})\)

A state transition matrix for the local coordinate vector \(\vec{z}\) can be derived by inverting the \(\Psi\) matrix defined in equation (4.13), and pre-multiplying this inverse first by the \(\Sigma\) matrix defined in equations (4.15) and (4.32) and then by the \(\Psi\) matrix. The resulting matrix relates small displacements in position and velocity at one epoch to those at another as a function of the constant primary orbit elements \(\alpha_{P1}\) and the true anomaly of the primary vehicle at each epoch \((f_{P2}, f_{P1})\) as:

\[
\Phi(f_{P2}, f_{P1}, \alpha_{P1}) = \Psi(f_{P2}, \alpha_{P1})\Sigma(f_{P2}, f_{P1}, \alpha_{P1})\Psi^{-1}(f_{P1}, \alpha_{P1})
\]

(4.33)

such that:

\[
\vec{z}(f_{P2}) = \Phi(f_{P2}, f_{P1}, \alpha_{P1}) \vec{z}(f_{P1})
\]

(4.34)

The 6 by 6 matrix \(\Psi(f_P, \alpha_{P1})\) can be analytically inverted rather easily because it has been partitioned into in-plane and out-of-plane components. Specific details of this algebra will be deleted for simplicity with only the resulting \(\Psi^{-1}(f_P, \alpha_{P1})\) matrix components given in table 4.3.
4.5 Comparison of Linearization Errors

A simulation was developed that computes the relative position and velocity of two satellites in proximate, highly elliptical orbits. Several test cases were used to verify the results of this simulation and bound the range of applicability for the state transition matrix method derived in section 4.4. Results of this simulation are compared directly with the integrated, nonlinear equations of motion for two individual orbits. The state transition matrix derivation is also compared directly with the integration of a set of linearized differential equations.

The state transition matrix method, described in detail in the previous sections, is propagated forward to each true anomaly point of the primary vehicle orbit. This is done with the transformation given by equation (4.34). Cartesian position and velocity in the local reference frame are then computed at this new orbital position through the \( \Psi \) matrix. The results from this state transition matrix approach will be identified by “Linear STM” in figure 4.4 and by “State Trans. Matrix” in figure 4.5.

The third set of results shows the integration of the linearized set of ordinary differential equations which describe the relative motion [Tschauer and Hempel, 1963], [Shulman and Scott, 1966]. These equations are analogous to Hill’s equations for the relative motion in a circular orbit.

\[
\frac{d^2 \tilde{x}}{df_P^2} - 2 \frac{d \tilde{y}}{df_P} = \frac{3 \tilde{x}}{1 + e_P \cos f_P} \tag{4.35}
\]

\[
\frac{d^2 \tilde{y}}{df_P^2} + 2 \frac{d \tilde{x}}{df_P} = 0 \tag{4.36}
\]

\[
\frac{d^2 \tilde{z}}{df_P^2} + \tilde{z} = 0 \tag{4.37}
\]

The local position of the secondary vehicle is non-dimensionalized by the orbital radius of the primary vehicle in equations (4.35), (4.36) and (4.37)

\( \tilde{x} \equiv x/r_P, \tilde{y} \equiv y/r_P, \tilde{z} \equiv z/r_P \).
This set is integrated with the same initial conditions as the truth model. These integrated solutions of the linearized differential equations are identified as "Linear ODEs" on the relevant plots.

The integration of the linear differential equations for the relative motion were shown to be nearly identical to those results generated by the method presented in this paper. Both methods provide linearized solutions to the nonlinear relative motion problem for highly elliptic orbits. If both are equivalent, then any differences between their predictions should only be of second order. Closed form solutions to these equations have been derived in previous work as a function of the local initial conditions [Shulman and Scott, 1966], but not as a function of the orbital element differences. The method derived in section 4.4, however, provides a closed form mapping of relative position and velocity in the local frame from one part of the orbit to another without the need for integration and with the added benefit of defining the difference in relative orbital elements.

In addition to the comparisons described above, two other separate analyses were performed to evaluate the validity of the state transition matrix method. First, an assessment of the effect of linearization error on the propagation was made in order to determine the range of vehicle separations for which this model remains valid.

Several test cases were used for comparison between the three models described above; integrating the full nonlinear differential equations; integrating the linearized differential equations; and computing the closed-form state transition matrix. The representative test case shown is a primary vehicle in geostationary transfer orbit. The orbital elements of this example are given below:
\[ a_P = 24.521 \text{km} \]
\[ e_P = 0.71950 \]
\[ i_P = 28.5^\circ \]
\[ \Omega_P = 0.0^\circ \]
\[ \omega_P = 30.0^\circ \]

Each comparison shown uses these elements for the primary orbit, with initial epoch at perigee (i.e. \( M_P = \phi_P = 0 \)). These elements were then "perturbed" about these reference conditions in order to generate a set of orbit elements for the secondary (active) spacecraft. The epoch conditions (the local coordinates of the secondary when the primary is at perigee) were then propagated forward using the state transition matrix model derived in this paper. Finally, the linearized differential equations (4.35), (4.36) and (4.37) were integrated from these same initial conditions. Each of the following two plots compare results between the three models all in local coordinates.

### 4.5.1 Out-of-Plane (z) Model Comparisons

As seen in equation (4.37) and in the \( \Psi \) and \( \Psi^{-1} \) matrices (see tables 4.2 and 4.3), the out-of-plane motion (z) is decoupled from in-plane motion (x - y) in the linear model. For this analysis, the inclination of the primary orbit was perturbed in order to generate a series of initial separations in the out-of-plane (z) direction only. Figure 4.4 shows the results for all three models given initial perturbations in the local z coordinate of 0.5 km, 5 km and 50 km respectively. As shown on the figure, both the state transition matrix formulation and integrating the linear differential equations match the integration of the full nonlinear model almost exactly for two complete orbits. This result was expected, because the out-of-plane motion described by both the linear and the nonlinear models is a harmonic oscillator with a period equal to the orbital period of the primary.
Figure 4.4: Out-of-Plane Model Comparison
4.5.2 In-Plane($x - y$) Model Comparisons

As shown in equations (4.35) and (4.36), the in-plane motion ($x - y$) is much more complex than the out-of-plane motion. This motion, however, can be modeled using the state transition matrix formulation, assuming that certain restrictions are placed on the magnitude of the initial relative separation between the two orbits. These restrictions are essentially a result of the linearization process. If the initial separation is greater than the order of 1% of the primary radius, the linear model results will diverge more quickly from the integrated nonlinear results and thus eventually become useless for the prediction of the actual separations. A more detailed analysis of this linearization error is provided later.

Figure 4.5 shows the in-plane($x - y$) and out-of-plane($z$) comparisons of the relative motion for the geostationary transfer orbit. The initial conditions are perturbed in order to generate an in-plane initial separation of approximately 0.5% of the range at perigee ($r_p$), or approximately 37 km total. The exact initial conditions in local coordinates are as follows:

\[
\begin{align*}
\text{radial} : & \quad x = 31.40 \text{ km} \quad u = 0.0043 \text{ km/sec} \\
\text{along-track} : & \quad y = 19.85 \text{ km} \quad v = -0.0712 \text{ km/sec} \\
\text{cross-track} : & \quad z = 0.62 \text{ km} \quad w = 0.0110 \text{ km/sec}
\end{align*}
\]

As before, these initial conditions were propagated forward using the state transition matrix formulation as well as the integrated linearized differential equations for two orbits and compared directly with the results from integrating the full nonlinear differential equations. In the figure, both the radial ($x$) and along-track ($y$) motions follow the nonlinear (truth) results very closely for almost two orbits. As the orbits progress, however, the linear models begin to diverge slightly, especially in the along-track direction. This consequently results in a divergence in the out-of-plane motion due to the difference in the orbital periods.
Figure 4.5: In-Plane Model Comparison
Of note is how both the state transition matrix formulation and the integrated linear differential equations track each other almost exactly, even after both diverge from the nonlinear solution. This result is again expected since both are linear approximations to the relative motion problem, but are just derived from different approaches. Thus, for a small enough relative separation, this linear model provides a viable method for propagating the relative position and velocity from one part of the primary orbit to another.

4.5.3 Linearization Error

The sensitivity to initial secondary vehicle locations in the primary orbit at perigee was investigated by varying the initial local coordinates of the secondary first in the radial (r) direction by 0.01\(r_p\) and 0.0001\(r_p\), in which \(r_p\) is the perigee radius to the primary vehicle (6878 km in this example). Next, a similar set of initial separations in the along-track (y) coordinate were simulated. In both of these cases, the initial relative velocity between the two vehicles is zero.

A summary of the results of these test cases is given in figures 4.6 and 4.7. In each of these plots, the root sum square (RSS) of the actual relative coordinates \((x, y, z)\) is normalized with the primary vehicle radius \((r_p)\) at each point and plotted on a logarithmic scale. The RSS of the error in the three coordinates between the state transition matrix and an iterative solution of Kepler's equation is also plotted on the same scale. Lines are added to figure 4.6 to indicate where the separation between the two vehicles exceeds 0.1\(r_p\). Intuition would suggest that a separation of this order of magnitude would set the upper bound on the applicability of the linear model.

An indication of a good linear model is that the error resulting from higher order terms is at least two orders of magnitude less than the quantity being computed. With this in mind, the second horizontal line on figure 4.6 indicates an
RSS error of 0.001r_p or two orders of magnitude less than the aforementioned 0.1r_p total separation. These two plots are indicated with symbols representing the error and solid lines representing the total (RSS) separation. As can be seen on figure 4.6, the “truth” curves cross the 0.1r_p line near where the respective “error” curve crosses the 0.001r_p line. This indicates that the linear model is valid within the range of what could be reasonably considered “small” vehicle separations (|Δ| < 0.1r_p) because the errors resulting from neglected higher order terms are two orders of magnitude less than the vehicle separations. For larger vehicle separations, figure 4.6 shows that the linear model is no longer valid.

Figure 4.7 is a similar plot to 4.6, but for initial secondary separations in y of 0.01r_p, and 0.0001r_p at perigee. This figure indicates a much greater sensitivity of the linear model to initial along-track separation. The model diverges before the first apogee crossing (near f = 100°) for initial y = 0.01r_p. The secular growth in the true separation (and consequently the error) can only be the result of an error in the estimate of Δa between the two orbits. This is thought to arise from the coupling between the radial and the along-track motion in which a y displacement does give rise to a true Δa orbit element difference. No such coupling exists in the linear model, so that any input of only y would generates a Δa = 0 and hence, the linear model does not show any secular divergence.

4.5.4 Linear Model Summary

The linearized equations of motion, expressed as a state transition matrix, is a sufficient approximation of the relative orbital motion for small vehicle separations and over short time periods. Such an approximate model was used in some preliminary design studies in this research, but was not found to be adequate for actual use in the navigation filter. All of the subsequent work on this filter uses the complete two-body dynamics model derived earlier in this chapter.
Figure 4.6. Separation and Linearization Error for Secondary Vehicle Radial Displacement of $x$ at Perigee
Figure 4.7. Separation and Linearization Error for Secondary Vehicle Radial Displacement of $y$ at Perigee
4.6 Orbital Perturbation Error

In order to evaluate the validity of the state transition matrix model with respect to integrated orbits using higher order (non two-body) dynamics, an analysis was performed [Garrison, et al., 1995] to evaluate the effect that perturbations have on the relative motion of two satellites in highly elliptic orbits. The purpose of this was to validate the model in actual orbital conditions that a multiple satellite system might encounter. For this analysis, the JPL model ASAP [Kwok, 1987] was used to apply perturbations to two separate orbits in an inertial frame and then difference the results and compare these with the two-body orbit separations in the local reference frame. [These ASAP simulations were provided by Thomas G. Gardner of the University of Colorado.] Figure 4.8 shows the results of this analysis. The dashed line on the figure indicates the separation of the satellites for two-body motion in the local reference frame as a function of the primary vehicle true anomaly. The solid line represents the RSS error in the separations caused by the addition of perturbations. These perturbations consist of gravitational terms out to $L = 5$ and $M = 4$, lunar and solar third body disturbances, drag, and solar-radiation pressure. As shown in the figure, the initial separation of the vehicles is approximately 200 km, which is well within the range of the linearization restriction for this orbit as described above. As the orbit progresses, the error due to the perturbations grows but is never more than approximately 1% of the total separation. This result is significant in that the error due to perturbations is small as compared to the error inherent in linearization over the orbital time frames of interest.
Figure 4.8: Orbit Perturbation Error
CHAPTER 5
APPLICATION TO RELATIVE NAVIGATION

This chapter describes the specific details of the application of two-step filters to the problem of relative navigation in an elliptical orbit. A filter using a first step state defined by augmenting the second step state vector with the observation equation and a filter using first and second step state vectors that have the same dimensions are both derived. This chapter also provides the necessary expressions for application of the IEKF and the coordinate transformation filter to the same problem. Finally, two new solutions to the problem of properly initializing the first step state and its covariance are provided.

5.1 Basic Assumptions

The focus of this study is the comparison of different nonlinear estimation techniques to the design of a relative navigation filter. The estimation is performed using only measurements of relative position and velocity and estimating only relative states. Solution of the complete navigation problem, however, requires determination of not only the six relative states but the orbit of the primary vehicle as well. Attitude determination of the primary may or may not be a required output, depending upon the application and the control requirements on the primary. In any case, attitude knowledge is necessary for the proper incorporation of the elevation and azimuth measurements. In this study, however, perfect attitude and orbit knowledge for the primary vehicle are assumed. The relationship between the relative navigation filter to be developed and the external orbit and attitude determination filters is illustrated in figure 5.1.
Figure 5.1. Relationship Between the Relative Navigation Filter, the Orbit Determination Filter and the Attitude Determination Filter
5.2 Dynamic Model

The relative navigation filter estimates a six element state vector consisting of the differences in inertial position and velocity between the primary vehicle and the secondary vehicle. The Earth-centered inertial coordinate frame that these are measured in is identical to that used for the truth model in chapter 4 (see figure 4.1). X and Y are in the equatorial plane of the Earth with X defined along the ascending node of the primary vehicle's orbit. Z is parallel to the Earth's axis of rotation. The filter dynamic model states are defined in terms of the truth model variables as follows:

\[
\mathbf{\dot{x}} = \begin{bmatrix}
\frac{X_s - X_p}{a_p} & \frac{Y_s - Y_p}{a_p} & \frac{Z_s - Z_p}{a_p} & \frac{\dot{X}_s - \dot{X}_p}{\sqrt{\mu/a_p}} & \frac{\dot{Y}_s - \dot{Y}_p}{\sqrt{\mu/a_p}} & \frac{\dot{Z}_s - \dot{Z}_p}{\sqrt{\mu/a_p}}
\end{bmatrix}
\] (5.1)

The relative position states are scaled by a reference semimajor axis value and the relative velocities are scaled by a reference circular velocity. For all of these simulations, the (assumed constant) orbit elements of the primary vehicle are used for these scale factors. This was done to reduce scaling problems which may arise in performing the iterative solution on dimensional states that are orders of magnitude different in absolute value. It also allows some dimensional terms to cancel out, simplifying the filter equations. For this study it is assumed that the orbit of the primary is known to a high precision and the orbit elements of this vehicle will be used as constant parameters of the filter. The filter dynamic model further assumes that both the primary and the secondary are in two-body orbits.

The validity of two-body dynamics for the relative motion problem was discussed in chapter 4. The primary vehicle orbit is therefore defined by the constant Keplerian orbit elements; semimajor axis \((a_p)\); eccentricity \((e_p)\); argument of perigee \((\omega_p)\); and inclination \((i_p)\). In the inertial coordinate system defined earlier and in chapter 4, the right ascension of the ascending node is zero \((\Omega_p \equiv 0)\). True
anomaly of the primary \((f_P)\) is used as the independent variable. This eliminates
the need to separately propagate the primary vehicle orbit.

The dynamic model for the states is

\[
\frac{d\vec{x}}{df_P} = \frac{(1-e_P^2)^{3/2}}{(1+e_P \cos f_P)^2} \begin{bmatrix}
  x_4 \\
  x_5 \\
  x_6 \\
  \frac{\dot{x}_P(f_P)+x_1}{R_S(f_P)^3} + \frac{\dot{x}_P(f_P)}{R_P(f_P)^3} \\
  \frac{\dot{y}_P(f_P)+x_2}{R_S(f_P)^3} + \frac{\dot{y}_P(f_P)}{R_P(f_P)^3} \\
  \frac{\dot{z}_P(f_P)+x_3}{R_S(f_P)^3} + \frac{\dot{z}_P(f_P)}{R_P(f_P)^3}
\end{bmatrix}
\] (5.2)

The non-dimensional (scaled with \(a_P\)) primary vehicle position is given in
terms of the four constant orbit elements and true anomaly.

\[
\tilde{X}_P \equiv \frac{X_P}{a_P} = \frac{(1-e_P^2)}{(1+e_P \cos f_P)} \cos(\omega_P + f_P) 
\] (5.3)

\[
\tilde{Y}_P \equiv \frac{Y_P}{a_P} = \frac{(1-e_P^2)}{(1+e_P \cos f_P)} \cos i_P \sin(\omega_P + f_P) 
\] (5.4)

\[
\tilde{Z}_P \equiv \frac{Z_P}{a_P} = \frac{(1-e_P^2)}{(1+e_P \cos f_P)} \sin i_P \sin(\omega_P + f_P) 
\] (5.5)

The Earth-centered radius to the primary and the secondary (also scaled with \(a_P\))
are given by

\[
\tilde{R}_P = \frac{1-e_P^2}{1+e_P \cos f_P} 
\] (5.6)

\[
\tilde{R}_T = \sqrt{(\tilde{X}_P + x_1)^2 + (\tilde{Y}_P + x_2)^2 + (\tilde{Z}_P + x_3)^2} 
\] (5.7)

The partial derivative matrix of the dynamic equation, which is used to
propagate the state covariance matrix between estimates as \(\dot{\Phi} = A(x,t)\Phi\) is

\[
A(x,f_P) = \frac{(1-e_P^2)^{3/2}}{(1+e_P \cos f_P)^2} \tilde{A}
\]

in which the elements of \(\tilde{A}\) are given in table 5.1.
Table 5.1: Elements of $\tilde{A}$ for the State Transition Matrix Integration.

\[
\begin{align*}
\tilde{A}_{1,1} &= 0 \quad \tilde{A}_{3,1} = \frac{3(X_p + z_1)^2 - R_s^2}{R_s^2} \\
\tilde{A}_{1,2} &= 0 \quad \tilde{A}_{4,2} = \frac{3(Y_p + z_2)(Y_p + z_2)}{R_s^2} \\
\tilde{A}_{1,3} &= 0 \quad \tilde{A}_{4,3} = \frac{3(Y_p + z_2)(Z_p + z_3)}{R_s^2} \\
\tilde{A}_{1,4} &= 1 \quad \tilde{A}_{4,4} = 0 \\
\tilde{A}_{1,5} &= 0 \quad \tilde{A}_{4,5} = 0 \\
\tilde{A}_{1,6} &= 0 \quad \tilde{A}_{4,6} = 0 \\
\tilde{A}_{2,1} &= 0 \quad \tilde{A}_{5,1} = \frac{3(X_p + z_1)(Y_p + z_2)}{R_s^2} \\
\tilde{A}_{2,2} &= 0 \quad \tilde{A}_{5,2} = \frac{3(Y_p + z_2)^2 - R_s^2}{R_s^2} \\
\tilde{A}_{2,3} &= 0 \quad \tilde{A}_{5,3} = \frac{3(Y_p + z_2)(Z_p + z_3)}{R_s^2} \\
\tilde{A}_{2,4} &= 0 \quad \tilde{A}_{5,4} = 0 \\
\tilde{A}_{2,5} &= 1 \quad \tilde{A}_{5,5} = 0 \\
\tilde{A}_{2,6} &= 0 \quad \tilde{A}_{5,6} = 0 \\
\tilde{A}_{3,1} &= 0 \quad \tilde{A}_{6,1} = \frac{3(X_p + z_1)(Z_p + z_3)}{R_s^2} \\
\tilde{A}_{3,2} &= 0 \quad \tilde{A}_{6,2} = \frac{3(Y_p + z_2)(Z_p + z_3)}{R_s^2} \\
\tilde{A}_{3,3} &= 0 \quad \tilde{A}_{6,3} = \frac{3(Z_p + z_3)^2 - R_s^2}{R_s^2} \\
\tilde{A}_{3,4} &= 0 \quad \tilde{A}_{6,4} = 0 \\
\tilde{A}_{3,5} &= 0 \quad \tilde{A}_{6,5} = 0 \\
\tilde{A}_{3,6} &= 1 \quad \tilde{A}_{6,6} = 0
\end{align*}
\]
5.3 Observations and First Step States

In this section, the mechanization models for two different measurement sets are defined. The corresponding first step states for the two-step filter are defined here as well. It should be noted that the observations and the first step states are also scaled exactly the same as for the second step states in the definition in equation (5.1). Range is scaled by the primary semimajor axis and range rate is scaled by the primary circular velocity.

5.3.1 Range, Range Rate and Angle Measurements

The first set of measurements considered are those of range, range rate, and the direction cosines. This set of measurements makes the state estimation very observable and the filter is not strongly dependent upon the dynamic model to incorporate measurements over a long arc of data. This set of observations eliminates problems with weakly observable out of plane errors for nearly coplanar orbits. It also prevents some possible ambiguities in which the iterations converge to the wrong orbit when using only range or range rate measurements.

A description of the radar based hardware and measurement model are provided in chapter 4. Relative angles are expressed in the orbit local system as described in chapter 4 (see figure 4.2). With this set of measurements defined in chapter 4, the nonlinear observation equation is

\[
\begin{align*}
\rho & \\
\frac{x_1 \sin \tilde{\omega} + x_2 \sin \tilde{\omega} + x_3 \sin \tilde{\omega}}{\rho} & \\
\frac{1}{\rho} (x_1 \cos \tilde{\omega} - x_2 \sin \tilde{\omega}) & \\
\frac{1}{\rho} B
\end{align*}
\]

(5.8)

The substitutions \( \rho = \sqrt{x_1^2 + x_2^2 + x_3^2} \) and

\[
B = x_1 \cos (\omega + f_p) + x_2 \cos \omega \sin (\omega + f_p) + x_3 \sin \omega \sin (\omega + f_p)
\]

have been made and will be used for the remainder of this chapter.
The elements of the linearized observation equation for the IEKF are given in Table 5.2.

The first step state vector is formed by augmenting the second step states with the measurement equation. This is trivially obtained from Equation (5.8).

\[
\tilde{y} = f(\tilde{x}) = \begin{bmatrix} \rho \\ \frac{x_1 e_1 + x_2 e_2 + x_3 e_3}{\rho} \\ \frac{1}{\rho}(x_3 \cos i \rho - x_2 \sin i \rho) \\ \frac{1}{\rho} B \\ x_1 \\ \vdots \\ x_6 \end{bmatrix}
\]  

(5.9)

The partial derivative matrix contains elements of the linearized observation matrix in Table 5.2.

\[
\frac{\partial f}{\partial x} = \begin{bmatrix} H_{1,1} & H_{1,2} & \cdots & H_{1,6} \\ H_{2,1} & H_{2,2} & \cdots & H_{2,6} \\ H_{3,1} & H_{3,2} & \cdots & H_{3,6} \\ H_{4,1} & H_{4,2} & \cdots & H_{4,6} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}
\]  

(5.10)

A two-step filter using the second step state vector augmented by the measurement equation has an observation matrix of the form

\[
H = [I \ 0]
\]  

(5.11)

To construct a two-step filter with first and second step states that have the
Table 5.2: Linearized Measurement Matrix to be Used in the IEKF

\[
\begin{align*}
H_{1,1} &= \frac{\xi_1}{\rho} \\
H_{1,3} &= \frac{\xi_3}{\rho} \\
H_{1,5} &= 0 \\
H_{2,1} &= \frac{\xi_1}{\rho^2}(x_1x_4 + x_2x_5 + x_3x_6) \\
H_{2,3} &= \frac{\xi_3}{\rho^2}(x_1x_4 + x_2x_5 + x_3x_6) \\
H_{2,5} &= \frac{\xi_3}{\rho} \\
H_{3,1} &= \frac{\xi_3}{\rho^2}(x_2 \sin i_\rho - x_3 \cos i_\rho) \\
H_{3,3} &= \frac{\xi_3}{\rho^2}(x_2 \sin i_\rho - x_3 \cos i_\rho) + \frac{\cos i_\rho}{\rho} \\
H_{3,5} &= 0 \\
H_{4,1} &= -\frac{\xi_3}{\rho^2}R + \frac{\cos(\omega p + f_p)}{\rho} \\
H_{4,3} &= -\frac{\xi_3}{\rho^2}R + \frac{\sin i_\rho \sin(\omega p + f_p)}{\rho} \\
H_{4,5} &= 0 \\
H_{1,2} &= \frac{\xi_2}{\rho} \\
H_{1,4} &= 0 \\
H_{1,6} &= 0 \\
H_{2,2} &= \frac{\xi_2}{\rho^2}(x_1x_4 + x_2x_5 + x_3x_6) \\
H_{2,4} &= \frac{\xi_2}{\rho} \\
H_{2,6} &= \frac{\xi_2}{\rho} \\
H_{3,2} &= \frac{\xi_2}{\rho^2}(x_2 \sin i_\rho - x_3 \cos i_\rho) - \frac{\sin i_\rho}{\rho} \\
H_{3,4} &= 0 \\
H_{3,6} &= 0 \\
H_{4,2} &= -\frac{\xi_2}{\rho^2}R + \frac{\cos i_\rho \sin(\omega p + f_p)}{\rho} \\
H_{4,4} &= 0 \\
H_{4,6} &= 0
\end{align*}
\]
same dimension, the following first step state vector is defined:

\[
\hat{y} = f(\tilde{x}) = \begin{bmatrix}
\frac{\dot{x}_1 x_4}{\rho} \\
\frac{\dot{x}_2 x_5}{\rho} \\
\frac{\dot{x}_2 x_6}{\rho} \\
\rho \\
\frac{1}{\rho}(x_3 \cos i_\rho - x_2 \sin i_\rho) \\
\frac{1}{\rho}B
\end{bmatrix}
\] (5.12)

The partial derivative matrix for this filter is given in Table 5.3.

For these first step states, the observation matrix is:

\[
H = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\] (5.13)

5.3.2 Range Rate Measurements Only

For a comparison to the previous filter which used a large-dimensioned state vector, a filter processing a single scalar measurement of range rate is now considered. In this filter, the observation equation is mechanized as

\[
h(\tilde{x}) = \frac{x_1 x_4 + x_2 x_5 + x_3 x_6}{\sqrt{x_1^2 + x_2^2 + x_3^2}}
\] (5.14)

and the linearized measurement matrix for the IEKF is

\[
H(\tilde{x}) = \begin{bmatrix}
\frac{\dot{x}_1}{\rho} - \frac{\dot{x}_4}{\rho^2}(x_1 x_4 + x_2 x_5 + x_3 x_6) \\
\frac{\dot{x}_2}{\rho} - \frac{\dot{x}_5}{\rho^2}(x_1 x_4 + x_2 x_5 + x_3 x_6) \\
\frac{\dot{x}_2}{\rho} - \frac{\dot{x}_6}{\rho^2}(x_1 x_4 + x_2 x_5 + x_3 x_6) \\
\frac{\dot{x}_1}{\rho} \\
\frac{\dot{x}_2}{\rho} \\
\frac{\dot{x}_3}{\rho}
\end{bmatrix}^T
\] (5.15)
Table 5.3: Partial Derivative Matrix for $n = m$ Two-Step Filter

\[
\begin{align*}
\frac{\partial f_1}{\partial x_1} &= \frac{e_3}{\rho} - \frac{e_3^2}{\rho^3} \\
\frac{\partial f_1}{\partial x_2} &= -\frac{e_3^2}{\rho^2} \\
\frac{\partial f_1}{\partial x_3} &= 0 \\
\frac{\partial f_1}{\partial x_4} &= \frac{e_3}{\rho} \\
\frac{\partial f_1}{\partial x_5} &= 0 \\
\frac{\partial f_1}{\partial x_6} &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial f_1}{\partial x_1} &= -\frac{e_3^2}{\rho^2} \\
\frac{\partial f_1}{\partial x_2} &= \frac{e_3}{\rho} - \frac{e_3^2}{\rho^3} \\
\frac{\partial f_1}{\partial x_3} &= 0 \\
\frac{\partial f_1}{\partial x_4} &= \frac{e_3}{\rho} \\
\frac{\partial f_1}{\partial x_5} &= 0 \\
\frac{\partial f_1}{\partial x_6} &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial f_1}{\partial x_1} &= -\frac{e_3^2}{\rho^3} \\
\frac{\partial f_1}{\partial x_2} &= \frac{e_3}{\rho} \\
\frac{\partial f_1}{\partial x_3} &= 0 \\
\frac{\partial f_1}{\partial x_4} &= \frac{e_3}{\rho} \\
\frac{\partial f_1}{\partial x_5} &= 0 \\
\frac{\partial f_1}{\partial x_6} &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial f_1}{\partial x_1} &= \frac{e_3}{\rho} \sin^2 \rho - e_3^2 \sin \rho + e_3 \cos \rho \\
\frac{\partial f_1}{\partial x_2} &= \frac{e_3}{\rho} \sin^2 \rho - e_3^2 \sin \rho + e_3 \cos \rho \\
\frac{\partial f_1}{\partial x_3} &= 0 \\
\frac{\partial f_1}{\partial x_4} &= \frac{e_3}{\rho} \sin^2 \rho - e_3^2 \sin \rho + e_3 \cos \rho \\
\frac{\partial f_1}{\partial x_5} &= 0 \\
\frac{\partial f_1}{\partial x_6} &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial f_1}{\partial x_1} &= \frac{e_3}{\rho} \cos^2 \rho - e_3^2 \cos \rho + e_3 \sin \rho \\
\frac{\partial f_1}{\partial x_2} &= \frac{e_3}{\rho} \cos^2 \rho - e_3^2 \cos \rho + e_3 \sin \rho \\
\frac{\partial f_1}{\partial x_3} &= 0 \\
\frac{\partial f_1}{\partial x_4} &= \frac{e_3}{\rho} \cos^2 \rho - e_3^2 \cos \rho + e_3 \sin \rho \\
\frac{\partial f_1}{\partial x_5} &= 0 \\
\frac{\partial f_1}{\partial x_6} &= 0
\end{align*}
\]

\[
\begin{align*}
\frac{\partial f_1}{\partial x_1} &= \frac{e_3}{\rho} \sin \rho \cos \rho - e_3^2 \sin \rho \cos \rho + e_3 \sin \rho \\
\frac{\partial f_1}{\partial x_2} &= \frac{e_3}{\rho} \sin \rho \cos \rho - e_3^2 \sin \rho \cos \rho + e_3 \sin \rho \\
\frac{\partial f_1}{\partial x_3} &= 0 \\
\frac{\partial f_1}{\partial x_4} &= \frac{e_3}{\rho} \sin \rho \cos \rho - e_3^2 \sin \rho \cos \rho + e_3 \sin \rho \\
\frac{\partial f_1}{\partial x_5} &= 0 \\
\frac{\partial f_1}{\partial x_6} &= 0
\end{align*}
\]
In the state-augmented two-step filter, the first step states are obtained by augmenting the second step states by the range rate measurement equation.

\[
\bar{y} = f(x) = \begin{bmatrix}
\frac{x_1 x_4 + x_2 x_5 + x_3 x_6}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\
x_1 \\
x_2 \\
\vdots \\
x_6
\end{bmatrix}
\] (5.16)

The partial derivative matrix for this set of first step states is therefore

\[
\begin{align*}
\frac{\partial f_1}{\partial x_1} &= \frac{\varepsilon_1}{\rho} - \frac{\varepsilon_2}{\rho^3}(x_1 x_4 + x_2 x_5 + x_3 x_6) \\
\frac{\partial f_1}{\partial x_2} &= \frac{\varepsilon_2}{\rho} - \frac{\varepsilon_3}{\rho^3}(x_1 x_4 + x_2 x_5 + x_3 x_6) \\
\frac{\partial f_1}{\partial x_3} &= \frac{\varepsilon_3}{\rho} - \frac{\varepsilon_4}{\rho^3}(x_1 x_4 + x_2 x_5 + x_3 x_6) \\
\frac{\partial f_1}{\partial x_4} &= \frac{\varepsilon_4}{\rho} \\
\frac{\partial f_1}{\partial x_5} &= \frac{\varepsilon_5}{\rho} \\
\frac{\partial f_1}{\partial x_6} &= \frac{\varepsilon_6}{\rho} \\
\frac{\partial f_1}{\partial x_k} &= \delta_{(j-1),k} \text{ for } j > 1
\end{align*}
\] (5.17)

For a two-step filter designed with a first step state vector of the same dimension as the second step state vector, the same set of states and partial derivatives as defined in the previous range-rate-angles example are used (equation (5.12) and table 5.3). The measurement matrix for the range rate is:

\[
H = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\] (5.18)

5.4 Filter Initialization

It is important that the first step state estimate and the first step state covariance be initialized properly in order to realize the full benefits of the two-step filter. The \textit{a priori} first step state and covariance are expressed as the expected
values

\[ \hat{y}_0(-) = \langle f(x_0(-)) \rangle \]  

(5.19)

and

\[ P_{y_0}(-) = \langle (f(x_0(-)) - \hat{y}_0(-))(f(x_0(-)) - \hat{y}_0(-))^T \rangle \]  

(5.20)

These are determined entirely from the corresponding second step state a priori estimate \( \hat{x}_0(-) \) used to initialize the filter and its probability distribution.

As discussed in chapter 2, the first step state covariance matrix is a key element in the performance of the two-step filter. This is especially true when there are an equal number of first and second step states. For a general problem, equations (5.19) and (5.20) cannot be evaluated in closed form. In [Kasdin, et al. 1997] it was suggested to use the second order expansion of \( P_y \) for this initialization. The second order expansion is perhaps the simplest closed form approximation for \( P_y \) in filters which have a larger dimension first step state than second step state. This is because the first order expansion of \( P_y \) is the expression \( (\partial f/\partial x)P_x(\partial f/\partial x)^T \) which is rank deficient and therefore cannot be used as a covariance matrix.

A Monte-Carlo method is one procedure to numerically estimate the initial state and covariance. The Monte-Carlo calculation of expected values is performed by some (large) number of iterations of the following:

1. Generate the \( i^{th} \) noisy second step state vector

\[ \tilde{x}_i = \hat{x}_0(-) + \tilde{\eta}_i \]  

(5.21)

In which \( \tilde{\eta}_i \) is taken from an ensemble of random numbers having the statistical properties

\[ \langle \tilde{\eta}\tilde{\eta}^T \rangle = P_{x_0} \]  

(5.22)

\[ \langle \tilde{\eta} \rangle = 0 \]  

(5.23)
The random number generator used for these simulations is the same one evaluated in section 4.2 and used to simulate measurement noise.

(2) Compute the true first step state vector for \( \bar{x}_i \).

\[ \bar{y}^* = f(\bar{x}, t_0) \]  

\[ (5.24) \]

(3) Add these instances to the accumulated sum of the state error and state squared error.

(4) Return to step 1 and repeat \( N \) times. \( (N \) is very large.\)

(5) When completed, compute the mean and the mean squared error of the first step state, \( \bar{y} \), giving the \textit{a priori} estimates and the covariance matrix.

\[ \bar{y}_0(-) = \frac{1}{N} \sum_{i=1}^{N} \bar{y}^i \]  

\[ (5.25) \]

\[ P_{y_0}(-) = \frac{1}{N} \sum_{i=1}^{N} (\bar{y}^i - \bar{y}_0(-))(\bar{y}^i - \bar{y}_0(-))^T \]  

\[ (5.26) \]

For the cases in which the initial state errors are assumed to be Gaussian and un-correlated, it is also possible as well as practical to numerically integrate the expected value for each element of the first step state estimate and covariance one at a time. Symbolically, the expected value of any scalar function, \( g(\bar{x}) \), of the state \( \bar{x} \) is

\[ \langle g(\bar{x}) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \ldots, x_6)p(x_1)p(x_2) \cdots p(x_6)dx_1dx_2\cdots dx_6 \]  

\[ (5.27) \]

Numerical integration of an expression such as (5.27) is practical for all of the formulations of the first step states described above because the velocity terms appear only linearly. This allows (using the assumption of uncorrelated initial states) factoring of the expected values of the velocity terms outside of the integral leaving only three as opposed to six numerical integrations.

The actual terms of (5.27) for the state augmented \( (n = 10) \) two-step filter are listed in table 5.4. The terms for the two-step filter with an equal number of
first and second step states \((n = m)\) are listed in table 5.5. These calculations are done starting with the \textit{a priori} second step state estimate, \(\hat{x}(-)\), and covariance, \(P_x(-)\) and then computing the expected values indicated by \(< >\) through numerical integration of equation (5.27). This results in the \textit{a priori} estimate of 
\(\hat{y}(-) = \langle \hat{y} \rangle\) and its covariance matrix, \(P_y(-)\).

It may not always be proper to assume that the six initial states are uncoupled. For those cases in which there are significant off diagonal elements of \(P_x\), a Monte Carlo method using a coordinate transformation is recommended.

Both methods were applied to the same problem for validation. It was found that both sets of numerical results agree with each other to within 10 \% when \(2 \times 10^7\) samples are used in the Monte Carlo simulation and the numerical integrations are performed over intervals from negative to positive 5 standard deviations. For the data presented in later chapters, initial conditions were obtained by numerical integration. The initial conditions used in scenario 1 were verified through use of the Monte Carlo method.
Table 5.4: State Augmented (n = 10) Two-Step Filter Initialization

A Priori First Step State

\[
\begin{align*}
\langle y_1 \rangle &= \langle \rho \rangle \\
\langle y_2 \rangle &= \dot{x}_4(-) \langle x_1/\rho \rangle + \dot{x}_5(-) \langle x_2/\rho \rangle + \dot{x}_6(-) \langle x_3/\rho \rangle \\
\langle y_3 \rangle &= \cos i \rho \langle x_1/\rho \rangle - \sin i \rho \langle x_2/\rho \rangle \\
\langle y_4 \rangle &= \cos(\omega P + f_P)\langle x_1/\rho \rangle + \cos i \rho \sin(\omega P + f_P)\langle x_2/\rho \rangle + \\
&\quad \sin i \rho \sin(\omega P + f_P)\langle x_3/\rho \rangle \\
\langle y_5 \rangle &= \dot{x}_1(-) \\
\langle y_6 \rangle &= \dot{x}_2(-) \\
\langle y_7 \rangle &= \dot{x}_3(-) \\
\langle y_8 \rangle &= \dot{x}_4(-) \\
\langle y_9 \rangle &= \dot{x}_5(-) \\
\langle y_{10} \rangle &= \dot{x}_6(-)
\end{align*}
\]

A Priori First Step State Covariance

\[
\begin{align*}
P_{y_{1,1}} &= (P_{x_{1,1}} + \dot{x}_1(-)^2) + (P_{x_{2,2}} + \dot{x}_2(-)^2) + (P_{x_{3,3}} + \dot{x}_3(-)^2) - \langle y_1 \rangle^2 \\
P_{y_{1,2}} &= \dot{x}_1(-)\dot{x}_4(-) + \dot{x}_2(-)\dot{x}_5(-) + \dot{x}_3(-)\dot{x}_6(-) - \langle y_1 \rangle \langle y_2 \rangle \\
P_{y_{1,3}} &= \cos i \rho \dot{x}_3(-) - \sin i \rho \dot{x}_3(-) - \langle y_1 \rangle \langle y_3 \rangle \\
P_{y_{1,4}} &= \cos(\omega P + f_P)\dot{x}_1(-) + \cos i \rho \sin(\omega P + f_P)\dot{x}_2(-) + \\
&\quad \sin i \rho \sin(\omega P + f_P)\dot{x}_3(-) - \langle y_1 \rangle \langle y_4 \rangle \\
P_{y_{1,5}} &= \langle x_1/\rho \rangle - \langle y_1 \rangle \langle y_5 \rangle \\
P_{y_{1,6}} &= \langle x_2/\rho \rangle - \langle y_1 \rangle \langle y_6 \rangle \\
P_{y_{1,7}} &= \langle x_3/\rho \rangle - \langle y_1 \rangle \langle y_7 \rangle \\
P_{y_{1,8}} &= 0 \\
P_{y_{1,9}} &= 0 \\
P_{y_{1,10}} &= 0 \\
P_{y_{2,2}} &= (P_{x_{4,4}} + \dot{x}_4(-)^2)\langle x_1^2/\rho^2 \rangle + (P_{x_{5,5}} + \dot{x}_5(-)^2)\langle x_2^2/\rho^2 \rangle + \\
&\quad (P_{x_{6,6}} + \dot{x}_6(-)^2)\langle x_3^2/\rho^2 \rangle + 2\dot{x}_4(-)\dot{x}_5(-)\langle x_1x_2/\rho^2 \rangle + \\
&\quad 2\dot{x}_4(-)\dot{x}_6(-)\langle x_1x_3/\rho^2 \rangle + 2\dot{x}_5(-)\dot{x}_6(-)\langle x_2x_3/\rho^2 \rangle - \langle y_2 \rangle^2 \\
P_{y_{2,3}} &= \cos i \rho (\dot{x}_4(-)\langle x_1x_3/\rho \rangle + \dot{x}_5(-)(x_2x_3/\rho) + \dot{x}_6(-)\langle x_3^2/\rho \rangle) - \\
&\quad \sin i \rho (\dot{x}_4(-)\langle x_1x_2/\rho \rangle + \dot{x}_5(-)\langle x_2^2/\rho \rangle + \dot{x}_6(-)(x_2x_3/\rho)) - \langle y_2 \rangle \langle y_3 \rangle
\end{align*}
\]
Table 5.4. State Augmented ($n = 10$) Two-Step Filter Initialization (cont’d).

\[
P_{y_{2.4}} = \cos(\omega_P + f_P) \left( \dot{x}_4(-) \langle x_1^2/\rho^2 \rangle + \dot{x}_5(-) \langle x_1 x_2/\rho^2 \rangle + \dot{x}_6(-) \langle x_1 x_3/\rho^2 \rangle \right) + \cos i_P \sin(\omega_P + f_P) \times \
\left( \dot{x}_4(-) \langle x_1 x_2/\rho^2 \rangle + \dot{x}_5(-) \langle x_2^2/\rho^2 \rangle + \dot{x}_6(-) \langle x_2 x_3/\rho^2 \rangle \right) + \
\sin i_P \sin(\omega_P + f_P) \times \
\left( \dot{x}_4(-) \langle x_1 x_3/\rho^2 \rangle + \dot{x}_5(-) \langle x_2 x_3/\rho^2 \rangle + \dot{x}_6(-) \langle x_3^2/\rho^2 \rangle \right) - \langle y_2 \rangle \langle y_4 \rangle
\]

\[
P_{y_{2.5}} = \dot{x}_4(-) \langle x_1^2/\rho \rangle + \dot{x}_5(-) \langle x_1 x_2/\rho \rangle + \dot{x}_6(-) \langle x_1 x_3/\rho \rangle - \langle y_2 \rangle \langle y_5 \rangle
\]

\[
P_{y_{2.6}} = \dot{x}_4(-) \langle x_1 x_2/\rho \rangle + \dot{x}_5(-) \langle x_2^2/\rho \rangle + \dot{x}_6(-) \langle x_2 x_3/\rho \rangle - \langle y_2 \rangle \langle y_6 \rangle
\]

\[
P_{y_{2.7}} = \dot{x}_4(-) \langle x_1 x_3/\rho \rangle + \dot{x}_5(-) \langle x_2 x_3/\rho \rangle + \dot{x}_6(-) \langle x_3^2/\rho \rangle - \langle y_2 \rangle \langle y_7 \rangle
\]

\[
P_{y_{2.8}} = P_{x_{1.4}} \langle x_1/\rho \rangle
\]

\[
P_{y_{2.9}} = P_{x_{2.5}} \langle x_2/\rho \rangle
\]

\[
P_{y_{2.10}} = P_{x_{6.6}} \langle x_3/\rho \rangle
\]

\[
P_{y_{3.3}} = \cos^2 i_P \langle x_3^2/\rho^2 \rangle - 2 \sin i_P \cos i_P \langle x_2 x_3/\rho^2 \rangle + \
\sin^2 i_P \langle x_3^2/\rho^2 \rangle - \langle y_3 \rangle^2
\]

\[
P_{y_{3.4}} = \cos i_P \cos(\omega_P + f_P) \langle x_1 x_3/\rho^2 \rangle + \cos^2 i_P \sin(\omega_P + f_P) \langle x_2 x_3/\rho^2 \rangle + \
\sin i_P \cos i_P \sin(\omega_P + f_P) \langle x_3^2/\rho^2 \rangle - \sin i_P \cos(\omega_P + f_P) \langle x_1 x_2/\rho^2 \rangle - \
\sin i_P \cos i_P \sin(\omega_P + f_P) \langle x_2^2/\rho^2 \rangle - \sin^2 i_P \sin(\omega_P + f_P) \langle x_2 x_3/\rho^2 \rangle - \
\langle y_3 \rangle \langle y_4 \rangle
\]

\[
P_{y_{3.5}} = \cos i_P \langle x_1 x_3/\rho \rangle - \sin i_P \langle x_1 x_2/\rho \rangle - \langle y_3 \rangle \langle y_5 \rangle
\]

\[
P_{y_{3.6}} = \cos i_P \langle x_2 x_3/\rho \rangle - \sin i_P \langle x_2^2/\rho \rangle - \langle y_3 \rangle \langle y_6 \rangle
\]

\[
P_{y_{3.7}} = \cos i_P \langle x_3^2/\rho \rangle - \sin i_P \langle x_2 x_3/\rho \rangle - \langle y_3 \rangle \langle y_7 \rangle
\]

\[
P_{y_{3.8}} = 0
\]

\[
P_{y_{3.9}} = 0
\]

\[
P_{y_{3.10}} = 0
\]

\[
P_{y_{4.4}} = \cos^2(\omega_P + f_P) \langle x_1^2/\rho^2 \rangle + 2 \cos i_P \sin(\omega_P + f_P) \cos(\omega_P + f_P) \langle x_1 x_2/\rho^2 \rangle + \
2 \sin i_P \cos(\omega_P + f_P) \sin(\omega_P + f_P) \langle x_1 x_3/\rho^2 \rangle + \
\cos^2 i_P \sin^2(\omega_P + f_P) \langle x_2^2/\rho^2 \rangle + \
2 \sin i_P \cos i_P \sin^2(\omega_P + f_P) \langle x_2 x_3/\rho^2 \rangle + \sin^2 i_P \sin(\omega_P + f_P) \langle x_3^2/\rho^2 \rangle - \
\langle y_4 \rangle^2
\]
Table 5.4. State Augmented \((n = 10)\) Two-Step Filter Initialization (cont’d).

\[
P_{y4,5} = \cos(\omega_p + f_p) \langle x_1^2 / \rho \rangle + \cos i_p \sin(\omega_p + f_p) \langle x_1 x_2 / \rho \rangle + \\
sin i_p \sin(\omega_p + f_p) \langle x_1 x_3 / \rho \rangle - \langle y_4 \rangle \langle y_5 \rangle
\]

\[
P_{y4,6} = \cos(\omega_p + f_p) \langle x_1 x_2 / \rho \rangle + \cos i_p \sin(\omega_p + f_p) \langle x_2^2 / \rho \rangle + \\
sin i_p \sin(\omega_p + f_p) \langle x_1 x_3 / \rho \rangle - \langle y_4 \rangle \langle y_6 \rangle
\]

\[
P_{y4,7} = \cos(\omega_p + f_p) \langle x_1 x_3 / \rho \rangle + \cos i_p \sin(\omega_p + f_p) \langle x_2 x_3 / \rho \rangle + \\
sin i_p \sin(\omega_p + f_p) \langle x_3^2 / \rho \rangle - \langle y_4 \rangle \langle y_7 \rangle
\]

\[
P_{y4,8} = 0
\]

\[
P_{y4,9} = 0
\]

\[
P_{y4,10} = 0
\]

\[
P_{y_i,i} = P_{x_{i-4,i-4}} \text{ for } i \geq 5
\]

\[
P_{y_i,j} = 0 \text{ for } i \neq j \text{ and } i \geq 5
\]

The expected values; \(\langle \rho \rangle, \langle x_1 / \rho \rangle, \langle x_2 / \rho \rangle, \langle x_3 / \rho \rangle, \langle x_1^2 / \rho \rangle, \langle x_2^2 / \rho \rangle, \langle x_3^2 / \rho \rangle, \langle x_1 x_2 / \rho \rangle, \langle x_1 x_3 / \rho \rangle, \langle x_2 x_3 / \rho \rangle, \langle x_1^2 / \rho^2 \rangle, \langle x_2^2 / \rho^2 \rangle, \langle x_3^2 / \rho^2 \rangle, \langle x_1 x_2 / \rho^2 \rangle, \langle x_1 x_3 / \rho^2 \rangle, \langle x_2 x_3 / \rho^2 \rangle\) are computed by numerically integrating (5.27) and \(\rho = \sqrt{x_1^2 + x_2^2 + x_3^2}. \)
Table 5.5: Equal Dimensional \((n = m)\) Two-Step Filter Initialization

**A Priori First Step State**

\[
\begin{align*}
    \langle y_1 \rangle &= \hat{x}_1(-) \langle x_1 / \rho \rangle \\
    \langle y_2 \rangle &= \hat{x}_5(-) \langle x_2 / \rho \rangle \\
    \langle y_3 \rangle &= \hat{x}_6(-) \langle x_3 / \rho \rangle \\
    \langle y_4 \rangle &= \langle \rho \rangle \\
    \langle y_5 \rangle &= -\sin i \rho \langle x_2 / \rho \rangle + \cos i \rho \langle x_3 / \rho \rangle \\
    \langle y_6 \rangle &= \cos(\omega_P + f_P) \langle x_1 / \rho \rangle + \cos i \rho \langle \omega_P + f_P \rangle \langle x_2 / \rho \rangle + \\
    &\sin i \rho \langle \omega_P + f_P \rangle \langle x_3 / \rho \rangle
\end{align*}
\]

**A Priori First Step State Covariance Matrix**

\[
\begin{align*}
    P_{y_{1,1}} &= (P_{x_{1,1}} + \hat{x}_4(-)^2) \langle x_1^2 / \rho^2 \rangle - \langle y_1 \rangle^2 \\
    P_{y_{1,2}} &= \hat{x}_4(-)\hat{x}_5(-) \langle x_1 x_2 / \rho^2 \rangle - \langle y_1 \rangle \langle y_2 \rangle \\
    P_{y_{1,3}} &= \hat{x}_4(-)\hat{x}_6(-) \langle x_1 x_3 / \rho^2 \rangle - \langle y_1 \rangle \langle y_3 \rangle \\
    P_{y_{1,4}} &= \hat{x}_1(-)\hat{x}_4(-) - \langle y_1 \rangle \langle y_4 \rangle \\
    P_{y_{1,5}} &= -\hat{x}_4(-) \sin i \rho \langle x_1 x_2 / \rho^2 \rangle + \hat{x}_4(-) \cos i \rho \langle x_1 x_3 / \rho^2 \rangle - \langle y_1 \rangle \langle y_5 \rangle \\
    P_{y_{1,6}} &= \hat{x}_4(-) \cos(\omega_P + f_P) \langle x_1^2 / \rho^2 \rangle + \hat{x}_4(-) \cos i \rho \langle \omega_P + f_P \rangle \langle x_1 x_2 / \rho^2 \rangle + \\
    &\hat{x}_4(-) \sin i \rho \langle \omega_P + f_P \rangle \langle x_1 x_3 / \rho^2 \rangle - \langle y_1 \rangle \langle y_6 \rangle \\
    P_{y_{2,2}} &= (P_{x_{2,2}} + \hat{x}_5(-)^2) \langle x_2^2 / \rho^2 \rangle - \langle y_2 \rangle^2 \\
    P_{y_{2,3}} &= \hat{x}_5(-)\hat{x}_6(-) \langle x_2 x_3 / \rho^2 \rangle - \langle y_2 \rangle \langle y_3 \rangle \\
    P_{y_{2,4}} &= \hat{x}_2(-)\hat{x}_4(-) - \langle y_2 \rangle \langle y_4 \rangle \\
    P_{y_{2,5}} &= -\hat{x}_5(-) \sin i \rho \langle x_2^2 / \rho^2 \rangle + \hat{x}_5 \cos i \rho \langle x_2 x_3 / \rho^2 \rangle - \langle y_2 \rangle \langle y_5 \rangle \\
    P_{y_{2,6}} &= \hat{x}_5(-) \cos(\omega_P + f_P) \langle x_1^2 / \rho^2 \rangle + \hat{x}_5(-) \cos i \rho \langle \omega_P + f_P \rangle \langle x_2^2 / \rho^2 \rangle + \\
    &\hat{x}_5(-) \sin i \rho \langle \omega_P + f_P \rangle \langle x_2 x_3 / \rho^2 \rangle - \langle y_2 \rangle \langle y_6 \rangle \\
    P_{y_{3,3}} &= (P_{x_{3,3}} + \hat{x}_6(-)^2) \langle x_3^2 / \rho^2 \rangle - \langle y_3 \rangle^2 \\
    P_{y_{3,4}} &= \hat{x}_3(-)\hat{x}_6(-) - \langle y_3 \rangle \langle y_4 \rangle \\
    P_{y_{3,5}} &= -\hat{x}_6(-) \sin i \rho \langle x_2 x_3 / \rho^2 \rangle + \hat{x}_6(-) \cos i \rho \langle x_3^2 / \rho^2 \rangle - \langle y_3 \rangle \langle y_5 \rangle \\
    P_{y_{3,6}} &= \hat{x}_6(-) \cos(\omega_P + f_P) \langle x_1^2 / \rho^2 \rangle + \hat{x}_6(-) \cos i \rho \langle \omega_P + f_P \rangle \langle x_2 x_3 / \rho^2 \rangle + \\
    &\hat{x}_6(-) \sin i \rho \langle \omega_P + f_P \rangle \langle x_3^2 / \rho^2 \rangle - \langle y_3 \rangle \langle y_6 \rangle
\end{align*}
\]
Table 5.5 Equal Dimensional \((n = m)\) Two-Step Filter Initialization (cont’d).

\[ P_{y_4.4} = (P_{x_{1.1}} + \bar{x}_1(-)^2) + (P_{x_{2.2}} + \bar{x}_2(-)^2) + (P_{x_{3.3}} + \bar{x}_3(-)^2) - \langle y_4 \rangle^2 \]

\[ P_{y_4.5} = -\sin i_p \langle \bar{x}_2 - \langle y_4 \rangle \rangle \langle x_2/\rho \rangle + \cos i_p \langle \bar{x}_3 - \langle y_4 \rangle \rangle \langle x_3/\rho \rangle \]

\[ P_{y_4.6} = \bar{x}_1 \cos(\omega_p + f_P) + \bar{x}_2 \cos i_p \sin(\omega_p + f_P) + \bar{x}_3 \sin i_p \sin(\omega_p + f_P) - \langle y_4 \rangle \langle y_6 \rangle \]

\[ P_{y_5.5} = \sin^2 i_p \langle x_2^2/\rho^2 \rangle - 2 \sin i_p \cos i_p \langle x_2 x_3/\rho^3 \rangle + \cos^2 i_p \langle x_3^2/\rho^2 \rangle - \langle y_5 \rangle^2 \]

\[ P_{y_5.6} = -\sin i_p \cos(\omega_p + f_P) \langle x_1 x_2/\rho^2 \rangle - \sin i_p \cos i_p \sin(\omega_p + f_P) \langle x_2^2/\rho^2 \rangle - \sin^2 i_p \cos(\omega_p + f_P) \langle x_2 x_3/\rho^3 \rangle + \cos i_p \cos(\omega_p + f_P) \langle x_1 x_3/\rho^2 \rangle + \cos^2 i_p \sin(i_p \sin(\omega_p + f_P) \langle x_2 x_3/\rho^2 \rangle + \sin i_p \cos i_p \sin(\omega_p + f_P) \langle x_3^2/\rho^2 \rangle - \langle y_5 \rangle \langle y_6 \rangle \]

\[ P_{y_6.6} = \cos^2(\omega_p + f_P) \langle x_1^2/\rho^2 \rangle + 2 \cos i_p \sin(\omega_p + f_P) \cos(\omega_p + f_P) \langle x_1 x_2/\rho^2 \rangle + 2 \sin i_p \sin(\omega_p + f_P) \cos(\omega_p + f_P) \langle x_1 x_3/\rho^2 \rangle + \cos^2 i_p \sin^2(\omega_p + f_P) \langle x_2^2/\rho^2 \rangle + 2 \cos i_p \sin i_p \sin^2(\omega_p + f_P) \langle x_2 x_3/\rho^2 \rangle + \sin^2 i_p \sin^2(\omega_p + f_P) \langle x_3^2/\rho^2 \rangle - \langle y_6 \rangle^2 \]

The expected values; \(\langle p \rangle, \langle x_1/\rho \rangle, \langle x_2/\rho \rangle, \langle x_3/\rho \rangle, \langle x_1^2/\rho^2 \rangle, \langle x_2^2/\rho^2 \rangle, \langle x_3^2/\rho^2 \rangle, \langle x_1 x_2/\rho^2 \rangle, \langle x_1 x_3/\rho^2 \rangle, \langle x_2 x_3/\rho^2 \rangle\), are computed by numerically integrating (5.27) and \(\rho = \sqrt{x_1^2 + x_2^2 + x_3^2}\).
CHAPTER 6

NUMERICAL SIMULATION

In this chapter, the properties of the two-step filter derived in chapters 2 and 3 are demonstrated by applying it to the relative navigation problem outlined in chapters 4 and 5. Two scenarios are considered which best illustrate the advantages as well as potential problems of the two-step estimator. The first scenario is a filter processing measurements of range, range rate and angles, which is started with a large initial error. This first example is used to demonstrate the advantages of the two-step estimator with equal dimensioned first and second step states over a similar coordinate transformation filter. The second scenario uses the filter to process range rate measurements alone and is simulated using an ensemble of initial state errors.

6.1 First Scenario: Large Initial Error with Good Observability

This scenario processes measurements of range, range rate, and direction cosines. For one application of the two-step filter, the first step states are defined by augmenting the second step state vector with the measurement equation. This 10 component first step state vector is given in equation (5.9). Another two-step filter is derived using the first step state vector given in equation (5.12). This results in a first step state vector of equal dimension to the second step state vector. A coordinate transformation filter is similarly applied using those same first step states, in which the nonlinear change of variables defined by equation (5.12) is applied at each time step, following the procedure of equations (2.31) to (2.37). An IEKF is also constructed based on the equations in table 5.2 and equations (5.8).
The reference conditions and uncertainty are based on an ESA study of rendezvous and docking in geostationary transfer orbits [ESA, 1983]. The specific numbers are listed in table 6.1. Initially, the primary and secondary are assumed to lie in the same orbit except for a separation in true anomaly. The 1σ uncertainties stated in that study are derived from predicted launch vehicle dispersions following a period of ground-based tracking of the two vehicles independently. Those orbit element uncertainties are combined into root sum squared uncertainties in the Earth centered inertial reference frame of figure 4.1 and assumed to be uncorrelated in that reference frame. Those uncertainties are used to form the second step state covariance. The first step state covariance is computed by numerical integration of the expected value as described in chapter 5. The same method is also used to compute the expected value of the first step state vector, \( \dot{y}_0 \).

These values are listed in table 6.2 for the ten element (state augmented) two-step filter and table 6.3 for the six element two-step filter.

The simulations are run for 800 seconds, processing measurements once per second starting when the primary vehicle is at perigee. A value of \( \epsilon = 10^{-27} \) is added as in equation (3.34). This prevents any failures resulting from ill-conditioned \( P_f \) over the ensemble of trajectories generated in this simulation.

For the six state filter, however, this value of \( \epsilon \) does not completely prevent an ill-conditioned \( P_f \) from causing an large increase in error for the six state filter. It is found to be very difficult to properly set \( \epsilon \) for a filter with equal dimensional states. One possible reason for this is that, for a filter with an equal number of first and second step states, an ill-conditioned first step covariance only occurs for a (numerically) rank deficient partial derivative matrix, \( \partial f / \partial x \). When such a near-singularity occurs, it is not expected that simply adding a small positive diagonal matrix onto it would improve its condition number. If \( A \) is ill-conditioned, then so is \( A + \epsilon \) for small \( \epsilon \). It is necessary to skip points in order to avoid severe
Table 6.1. Reference Initial Conditions of Elliptical Orbit Relative Motion Problem

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary:</td>
<td></td>
</tr>
<tr>
<td>$a_p$</td>
<td>24371 km</td>
</tr>
<tr>
<td>$e_p$</td>
<td>0.7301</td>
</tr>
<tr>
<td>$\omega_p$</td>
<td>180 deg</td>
</tr>
<tr>
<td>$i_p$</td>
<td>8.0 deg</td>
</tr>
<tr>
<td>$f_p(t_0)$</td>
<td>0 deg</td>
</tr>
<tr>
<td>Secondary:</td>
<td></td>
</tr>
<tr>
<td>$f_T(t_0)$</td>
<td>$-0.1818$ deg</td>
</tr>
</tbody>
</table>

numerical problems as a result of poorly conditioned $P_y$. A tolerance of $10^{-3}$ is used to compute the numerical rank of $\partial f/\partial x$. If this rank drops below 6, the next 20 seconds of input data are skipped. Single points at which the rank of $P_y$ drops in the two-step filter (even in the presence of the $\epsilon f$ term) are also skipped.

A total of 50 simulated trajectories are generated in which each set of simulated noisy data is processed by all four filters. For the two filters which require an iterative solution, the ten state two-step filter and the IEKF, the same tolerances are used for each iterative solution. The iterations are stopped after a minimum of 10 iterations have occurred and when the largest change in any element of the state vector is less than $10^{-12}$ times the norm of the state vector. A maximum of 500 iterations are allowed in the two-step filter and a maximum of 200 are allowed for each measurement in the IEKF. This is to prevent infinite loops in the event of divergent solutions.

This scenario represents one in which there are a large number of different measurements available and hence the problem has good observability at each time step. All of the estimators are expected to do well. The two-step filter, however, demonstrates better performance when the set of simulated runs are started from a large initial error. The initial conditions used in the truth model for all of these simulate trajectories are in error from the reference a priori state used to initialize the filter (which is given in table 6.1). These errors in inertial coordinates are as
follows ($\sigma$ is the standard error from the diagonals of the covariance matrix $P_2$):

\[
\begin{align*}
X &= 1100 \text{ m} \quad (0.7\sigma) \\
Y &= -25000 \text{ m} \quad (-2.0\sigma) \\
Z &= -1030 \text{ m} \quad (-0.6\sigma) \\
U &= -11.7 \text{ m/s} \quad (-0.9\sigma) \\
V &= 3 \text{ m/s} \quad (2.2\sigma) \\
W &= 0.1 \text{ m/s} \quad (0.6\sigma)
\end{align*}
\]

This has a root-sum-square magnitude of $3.3\sigma$.

Numerical results showing the mean position errors for the four different filters over the first 120 seconds of the simulation are given in figure 6.1. This figure also plots the bounds of $\pm \sigma/\sqrt{N}$ in which $N$ is the number of measurements used in the average. Those bounds are the standard deviation of the mean [Papoulis 1982]. Most of the two-step filter simulation state history lies within these bounds and is thus considered to be "unbiased" within the statistical significance of this simulation. The exception is the velocity error in the $W$ direction. Note, however, that the magnitude of this error is smaller than that in the other two directions. It does eventually decrease below the covariance bounds later in the simulation. Figure 6.2 is a similar plot of the velocity errors for the four filters. Whereas these plots demonstrate that all four filters converge for a large initial error and perform rather well, the two-step filter in both forms is better at removing large initial errors. This further confirms an assumption in the analysis of chapter 3, that the steady state covariance propagation and update of the two-step filter is approximately the same as that of a Kalman filter operating on the linearized model.

Comparison of the respective curves on these figures demonstrates that the two-step filter reduces initial error faster than the coordinate transformed filter when using the the same set of states. The differences are most significant for the
velocity states. This can be explained by the fact that three of the four observations; range, and the two direction cosines, are direct measurements of position. Range rate is the only direct measurement of velocity. The velocity state estimate is therefore more dependent upon the filter to propagate it from one measurement to the next.

The jumps in the plots for the coordinate transformation filter and the \( n = m \) two-step filter are the results of skipping measurements when \( \partial f/\partial x \) drops rank.

The comparison between the predicted covariance and the mean sum squared error for the filters is given in figures 6.3 and 6.4. The predicted covariance from the two-step filter is indicated by the solid line. That figure confirms the position and velocity plots in figures 6.1 and 6.2. The symbols on those plots for the filter mean squared error are the same as those used in figure 6.1 and 6.2. Again, note that the improvement in velocity error is larger than that in position error. The plots in figure 6.1 and 6.2 illustrate the relative performance of the different estimators compared in terms of removing large initial state errors. Figures 6.3 and 6.4 compare each filter estimation error with the prediction of that error.

The steady state estimation error, or bias, produced by each filter is compared in table 6.4. This table lists the root sum square of the three mean position and velocity errors, at points 400 seconds and 800 seconds into the simulation. The mean is only computed when a state estimate is available for all 50 trajectories in the ensemble (only points in which no measurements were deleted as the result of low rank \( \partial f/\partial x \) are plotted). For comparison, the standard deviation of the mean is also given in that table as a criterion for the statistical significance of the biases. This value is used to set the 95% confidence interval [Papoulis 1982]. From these results it can be inferred that only the CTF has a statistically significant bias at 400 seconds and that all of the filters are unbiased after 800
Figure 6.1: Mean Inertial Position Error for First Scenario
Figure 6.2: Mean Inertial Velocity Error for First Scenario
Table 6.2. Two-Step \( n = 10 \) Filter Initialization Data for Range-Rate-Angles Measurements

**First Step Covariance Matrix: \( P_y \)**

(units are derived from meters and seconds)

\[
\begin{bmatrix}
1.5 \times 10^8 & -4.0 \times 10^2 & 4.4 \times 10^2 & 7.9 \times 10^0 & 8.4 \times 10^3 & -1.5 \times 10^8 \\
-4.0 \times 10^2 & 4.5 \times 10^0 & -1.4 \times 10^{-3} & -5.4 \times 10^{-2} & 2.7 \times 10^3 & 4.1 \times 10^2 \\
4.1 \times 10^2 & -1.4 \times 10^{-3} & 2.7 \times 10^{-3} & 2.8 \times 10^{-5} & 2.7 \times 10^{-2} & -4.6 \times 10^2 \\
7.9 \times 10^0 & -5.4 \times 10^{-2} & 2.8 \times 10^{-5} & 1.1 \times 10^{-3} & -5.3 \times 10^4 & -8.1 \times 10^0 \\
8.4 \times 10^3 & 2.7 \times 10^3 & 2.7 \times 10^{-2} & -5.3 \times 10^1 & 2.8 \times 10^6 & 0 \\
-1.5 \times 10^8 & 4.1 \times 10^2 & -4.6 \times 10^2 & -8.1 \times 10^0 & 0 & 1.5 \times 10^8 \\
-4.3 \times 10^5 & 1.2 \times 10^6 & 5.5 \times 10^1 & -3.6 \times 10^{-2} & 0 & 0 \\
-3.6 \times 10^1 & 4.4 \times 10^{-1} & 1.6 \times 10^{-5} & -1.1 \times 10^{-4} & 0 & 0 \\
7.3 \times 10^0 & -1.7 \times 10^0 & 4.9 \times 10^{-5} & -9.2 \times 10^{-6} & 0 & 0 \\
-7.7 \times 10^{-2} & -6.4 \times 10^{-3} & 1.8 \times 10^{-6} & -3.0 \times 10^{-6} & 0 & 0 \\
\end{bmatrix}
\]

**Diagonals of the Second Step Covariance Matrix: \( P_x \)**

\[
\begin{bmatrix}
-4.3 \times 10^5 & -3.6 \times 10^1 & 7.3 \times 10^0 & -7.7 \times 10^{-2} \\
1.2 \times 10^0 & 4.4 \times 10^{-1} & -1.7 \times 10^0 & -6.4 \times 10^{-3} \\
5.5 \times 10^1 & 1.6 \times 10^{-5} & 4.9 \times 10^{-5} & 1.8 \times 10^{-6} \\
-3.6 \times 10^{-2} & -1.1 \times 10^{-4} & -9.2 \times 10^{-6} & -3.0 \times 10^{-6} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
2.9 \times 10^6 & 0 & 0 & 0 \\
0 & 1.7 \times 10^2 & 0 & 0 \\
0 & 0 & 1.7 \times 10^0 & 0 \\
0 & 0 & 0 & 4.4 \times 10^{-2} \\
\end{bmatrix}
\]

\[
\begin{align*}
2.8 \times 10^6 \text{ m}^2 \\
1.6 \times 10^8 \text{ m}^2 \\
3.0 \times 10^6 \text{ m}^2 \\
170.3 \text{ m}^2/\text{s}^2 \\
1.9 \text{ m}^2/\text{s}^2 \\
4.4 \times 10^{-2} \text{ m}^2/\text{s}^2
\end{align*}
\]
Table 6.3. Two-Step $n = 6$ Filter Initialization Data for Range-Rate-Angles Measurements

First Step Covariance Matrix: $P_y$
(units are derived from meters and seconds)

\[
\begin{bmatrix}
2.9 \times 10^0 & -7.2 \times 10^{-5} & -5.1 \times 10^{-5} & -3.9 \times 10^2 & -1.3 \times 10^{-3} & -5.4 \times 10^{-2} \\
-7.2 \times 10^{-5} & 1.6 \times 10^0 & 6.8 \times 10^{-6} & -1.0 \times 10^1 & -8.3 \times 10^{-5} & 4.4 \times 10^{-6} \\
-5.1 \times 10^{-5} & 6.8 \times 10^{-6} & 1.1 \times 10^{-3} & 1.3 \times 10^1 & 7.7 \times 10^{-5} & 1.0 \times 10^{-6} \\
-3.9 \times 10^2 & -1.0 \times 10^1 & 1.3 \times 10^1 & 1.5 \times 10^8 & 4.4 \times 10^2 & 7.7 \times 10^0 \\
-1.3 \times 10^{-3} & -8.3 \times 10^{-5} & 7.7 \times 10^{-5} & 4.4 \times 10^2 & 2.7 \times 10^{-3} & 2.7 \times 10^{-5} \\
-5.4 \times 10^{-2} & 4.4 \times 10^{-6} & 1.0 \times 10^{-6} & 7.7 \times 10^0 & 2.7 \times 10^{-5} & 1.1 \times 10^{-3}
\end{bmatrix}
\]

seconds.

For problems in which the measurement set is large and available frequently, the primary advantage of the two-step estimator is in the removal of large initial errors, which fall outside of the covariance bounds assumed for the filter.

One further comparison between the two-step filters, the IEKF and the coordinate transformation filters is shown in Table 6.5. The mean execution times for another complete set of 3000 second simulations of each filter are compared. The actual computation times for the simulated filters are meaningless by themselves because any real-time application would be done on a dedicated computer using software optimized for that specific purpose. For this reason, the numbers given in Table 6.5 are all normalized with the ten state two-step filter (defined as 1). The lower numbers for the coordinate transformation and the six state two-step filter are expected when one considers that in those filters the second step states are obtained by a closed form equation and no iteration is required.

The higher numbers for the IEKF are due to the fact that a UD covariance factorization filter is used. That algorithm, described in Appendix A, requires scalar measurements. Hence each of the four observations present at each time step must be processed individually. That requires four times the number of iterations
Figure 6.3: Mean Square Inertial Position Error for First Scenario
Figure 6.4: Mean Square Inertial Velocity Error for First Scenario
Table 6.4: State Estimation Biases for First Scenario

<table>
<thead>
<tr>
<th></th>
<th>Position t = 400 sec</th>
<th>Velocity t = 400 sec</th>
<th>Position t = 800 sec</th>
<th>Velocity t = 800 sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sqrt{E_{\theta}/\sqrt{N}}$</td>
<td>1.70 m</td>
<td>0.006 m/s</td>
<td>1.50 m</td>
<td>0.003 m/s</td>
</tr>
<tr>
<td>95% Confidence</td>
<td>3.34 m</td>
<td>0.012 m/s</td>
<td>2.95 m</td>
<td>0.005 m/s</td>
</tr>
<tr>
<td>Two-Step Filter</td>
<td>2.07 m</td>
<td>0.007 m/s</td>
<td>1.25 m</td>
<td>0.003 m/s</td>
</tr>
<tr>
<td>IEKF</td>
<td>2.76 m</td>
<td>0.009 m/s</td>
<td>0.87 m</td>
<td>0.003 m/s</td>
</tr>
<tr>
<td>Coordinate Change Filter</td>
<td>3.91 m</td>
<td>0.014 m/s</td>
<td>0.65 m</td>
<td>0.003 m/s</td>
</tr>
<tr>
<td>Two-Step Filter with n=m=6</td>
<td>2.96 m</td>
<td>0.010 m/s</td>
<td>0.98 m</td>
<td>0.003 m/s</td>
</tr>
</tbody>
</table>

as in the two-step filter. The iterations required in the IEKF, however, do not require the computation of the Hessian matrix or explicit evaluation of the cost function and thus involve fewer operations per each iteration.

6.2 Comparison between Two-Step Filter and A Coordinate Change Filter Using the Same set of States

In chapter 2 an analytical comparison was made between a two-step filter using the same dimension of first step and second step state vectors and a Kalman filter that uses a change of variables at each time step to make the measurement equation linear. It was argued there that the two-step filter is still expected to perform better than the coordinate change filter in spite of the similarities in the two filters. These results are now demonstrated numerically using the problem from the first scenario. A set of 50 Monte Carlo simulations are performed generating simulated noisy data which are used as observations for a two-step estimator and a coordinate transformation filter. Two other modified two-step filters were also compared. The first was initialized with a first step state estimate obtained directly from $\hat{y} = f(\hat{x})$ and a first step covariance computed the same as before, using a Monte Carlo integration of the expectation. The second filter was initialized with the correct initial value of $\hat{y}$ but a linearized approximation to the
Table 6.5: Comparison of Machine Execution Time: First Scenario

<table>
<thead>
<tr>
<th>Estimator</th>
<th>Mean Execution Time Normalized with Two-Step</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two-Step Filter</td>
<td>1</td>
</tr>
<tr>
<td>IEKF</td>
<td>1.5061</td>
</tr>
<tr>
<td>Coordinate Change Filter</td>
<td>0.3043</td>
</tr>
<tr>
<td>Two-Step Filter with n=m-6</td>
<td>0.3730</td>
</tr>
</tbody>
</table>

initial first step covariance matrix: \((\partial f/\partial x)P_z(\partial f/\partial x)^T\).

The results of these simulations are shown in figure 6.5 which is a plot of the mean position and velocity errors. Again, the mean is only computed when an estimate is available in all 50 trajectories in the ensemble. The important comparison is between the results for the two-step filters initialized with the approximate \(\hat{y}\) and \(P_y\). These approximations will initialize the first step state propagation and covariance propagation, respectively, to be the same as that for the coordinate change filter. It is clear from figure 6.5 that the two-step filter started with the linear covariance approximation gives almost identical results as the coordinate change filter. This confirms the argument given in chapter 2 that the improvement of the two-step filter over a simple coordinate change is in the approximation to the first step covariance. In the two-step filter this approximation is usually better because a smaller term is neglected at each time step. These results also imply that the realization of better performance in the two-step estimator requires proper initialization of the first step covariance matrix. The last comparison suggests that this is even more most important for filters with an equal number of first and second step states.
Figure 6.5. Comparison of a Six State Two-Step Filter with a Coordinate Change Filter
6.3 Second Scenario: Scalar Measurements with Poor Observability

The next series of simulations are presented for the filters processing a single scalar range rate measurement at each time step. A two-step filter using 7 first step states is created by state augmentation, as in equation (5.16). A two-step filter with a first and second step state vector of the same dimension is constructed which used the same states as those in the first scenario (5.12). The IEKF and coordinate transformation filters are also simulated for comparison.

Unlike the first scenario in which a large measurement set was used, the filter in this scenario will be heavily relied on to near-optimally combine the measurements and make the state observable. The problem of state observability is much worse for the reference mission described earlier in which both vehicles are in the same orbital plane. In this case the out of plane motion is very weakly observable. For such a case, some form of angle measurements is necessary for a robust navigation system. The objective at this time to to demonstrate the difference in estimation accuracy between the two-step filters and the IEKF and coordinate transformation filters. As such, an example orbit case which includes some out of plane motion as well as some radial motion was devised. This was formed by adding an additional inclination and eccentricity differences to the reference orbits in table 6.1. The scenario was started at primary true anomaly of \( f_P = -60^\circ \) and each simulation was run for 2000 points. A total of 50 Monte Carlo simulations are performed. The initial states are selected by adding randomly generated errors to the reference values in table 6.6. These errors are zero mean. Gaussian random numbers with statistics given by the second step state covariance matrix in tables 6.7. A discrete process noise term of \( Q_d = 4 \times 10^{-5} m^2/s^2 \) was used in the velocity states of the filter. No process noise was simulated in the truth model, however.

As discussed earlier, the addition of a small \( \epsilon \) to the first step state
Table 6.6. Reference Initial Conditions of Elliptical Orbit Relative Motion Problem

Range Rate Only

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$a_P$</td>
<td>= 24371 km</td>
<td></td>
</tr>
<tr>
<td>$e_P$</td>
<td>= 0.7301</td>
<td></td>
</tr>
<tr>
<td>$\omega_P$</td>
<td>= 180 deg.</td>
<td></td>
</tr>
<tr>
<td>$i_P$</td>
<td>= 8.0 deg.</td>
<td></td>
</tr>
<tr>
<td>$f_P(t_0)$</td>
<td>= -60 deg.</td>
<td></td>
</tr>
<tr>
<td>Secondary</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$X_S - X_P$</td>
<td>= 21414 m</td>
<td></td>
</tr>
<tr>
<td>$Y_S - Y_P$</td>
<td>= $-2.1637 \times 10^5$ m</td>
<td></td>
</tr>
<tr>
<td>$Z_S - Z_P$</td>
<td>= 93642 m</td>
<td></td>
</tr>
<tr>
<td>$U_S - U_P$</td>
<td>= $-10.5990$ m/s</td>
<td></td>
</tr>
<tr>
<td>$V_S - V_P$</td>
<td>= $-118.5730$ m/s</td>
<td></td>
</tr>
<tr>
<td>$W_S - W_P$</td>
<td>= $-147.7660$ m/s</td>
<td></td>
</tr>
</tbody>
</table>
Table 6.7: Two-Step $n = 7$ Filter Initialization Data for Range Measurements

First Step Covariance Matrix: $P_y$ (units are derived from meters and seconds)

\[
\begin{bmatrix}
1.0 \times 10^2 & -7.0 \times 10^3 & -5.5 \times 10^4 & -3.1 \times 10^3 & 5.6 \times 10^0 & -8.7 \times 10^1 \\
-7.0 \times 10^3 & 1.1 \times 10^8 & 0 & 0 & 0 \\
-5.5 \times 10^4 & 0 & 1.7 \times 10^8 & 0 & 0 \\
-3.1 \times 10^3 & 0 & 4.3 \times 10^6 & 0 & 0 \\
5.6 \times 10^0 & 0 & 0 & 6.2 \times 10^1 & 0 \\
-8.7 \times 10^1 & 0 & 0 & 0 & 9.5 \times 10^1 \\
9.2 \times 10^{-1} & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Diagonals of Second Step Covariance Matrix: $P_x$

1.10 $\times 10^8$ m$^2$
1.75 $\times 10^8$ m$^2$
4.35 $\times 10^6$ m$^2$
61.91 m$^2$/s$^2$
95.09 m$^2$/s$^2$
2.31 m$^2$/s$^2$

Table 6.8: Two-Step $n = 6$ Filter Initialization Data for Range Measurements

First Step Covariance Matrix: $P_y$ (units are derived from meters and seconds)

\[
\begin{bmatrix}
8.5 \times 10^{-1} & 2.5 \times 10^{-1} & 3.3 \times 10^{-2} & 1.5 \times 10^2 & 8.1 \times 10^{-5} & 8.5 \times 10^{-3} \\
2.5 \times 10^{-1} & 8.1 \times 10^1 & 3.6 \times 10^9 & 1.3 \times 10^4 & -2.2 \times 10^{-2} & -2.8 \times 10^{-3} \\
3.3 \times 10^{-2} & 3.6 \times 10^0 & 1.1 \times 10^1 & 3.5 \times 10^4 & -6.4 \times 10^{-2} & -3.3 \times 10^{-2} \\
1.5 \times 10^2 & 1.3 \times 10^4 & 3.5 \times 10^4 & 1.5 \times 10^8 & -2.2 \times 10^2 & -1.1 \times 10^2 \\
8.1 \times 10^{-5} & -2.2 \times 10^{-2} & -6.4 \times 10^{-2} & -2.2 \times 10^2 & 4.1 \times 10^{-4} & 2.2 \times 10^{-4} \\
8.5 \times 10^{-3} & -2.8 \times 10^{-3} & -3.3 \times 10^{-2} & -1.1 \times 10^2 & 2.2 \times 10^{-4} & 4.5 \times 10^{-4} \\
\end{bmatrix}
\]
Figure 6.6: Mean Inertial Position Error for Second Scenario
Figure 6.7: Mean Inertial Velocity Error for Second Scenario
Figure 6.8: Mean Inertial Position Squared Error for Second Scenario
Figure 6.9: Mean Inertial Velocity Squared Error for Second Scenario
covariance propagation in equation (2.53) was not always able to mitigate the
difficulties associated with rank deficiency of the partial derivative matrix itself.
The solution in this simulation was to compute the rank of this matrix at each time
step using a tolerance of $10^{-2}$. If the rank dropped at anytime then a set of 10
measurements are skipped. This works in most of the trajectories to prevent
undesirable numerical problems form an ill-conditioned $\partial f/\partial z$. However, if such a
low rank $\partial f/\partial z$ is encountered very early in the trajectory, then the filter has
difficulty getting properly started and large errors can be encountered. Within the
set of 50 randomly distributed trajectories simulated, 2 were found to experience
this problem for the $n = m$ two-step filter and one was found to experience it for
the CTF. These two divergent runs were easy to identify and were rejected from the
ensemble when used to compute and plot the mean error and mean squared errors.

The mean position and velocity error, indicating bounds of one standard
deviation predicted by the filter covariance matrix are plotted in figures 6.6 and
6.7. Again, two divergent runs caused by a low rank $\partial f/\partial z$ matrix at the start of
the simulation have been removed from the ensemble of the $n = m$ filter. One such
run has been removed from that for the CTF. Plots of the mean square error,
showing a comparison to the two-step filter predicted covariance are shown in
figure 6.8 and 6.9.

Note on these plots that the $n = m$ two-step filter and the CTF appear to
have very similar results. In fact, the state estimation error of each is comparable
to that for the seven state two-step filter. This appears to contradict the earlier
analysis which indicated that the two-step filter with an equal dimensioned first
and second step state vector would produce a better state estimate because it
makes a smaller approximation to the covariance update between the first and
second step states. For this specific case, however, the first step state covariance is
much closer to the approximation $(\partial f/\partial z) P_x (\partial f/\partial z)^T$ than that used in the first.
scenario. This is shown by comparing the difference between the initial $P_y$ and the linear approximation $(\partial f / \partial x) P_x (\partial f / \partial x)^T$ (which is essentially the term $G_0$ used in equation (2.56).

When the individual elements of the $P_y$ matrix are compared to those of the first order approximation this difference is significant. For the first scenario, the individual terms of the (normalized) difference

$$
\left[ P_y - \frac{\partial f}{\partial x} P_x \frac{\partial f}{\partial x}^T \right]_{i,j} / P_{y,i,j}
$$

are as follows:

<table>
<thead>
<tr>
<th></th>
<th>$2.1 \times 10^{-1}$</th>
<th>$1.7 \times 10^{0}$</th>
<th>$5.3 \times 10^{-1}$</th>
<th>$1.5 \times 10^{-1}$</th>
<th>$3.8 \times 10^{-1}$</th>
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<td>$1.7 \times 10^{0}$</td>
<td>$1.2 \times 10^{-1}$</td>
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<td>$1.5 \times 10^{-1}$</td>
</tr>
<tr>
<td>$1.5 \times 10^{-1}$</td>
<td>$-9.4 \times 10^{-2}$</td>
<td>$1.4 \times 10^{-1}$</td>
<td>$-7.6 \times 10^{-3}$</td>
<td>$1.5 \times 10^{-1}$</td>
<td>$1.5 \times 10^{-1}$</td>
<td>$1.5 \times 10^{-1}$</td>
</tr>
<tr>
<td>$3.8 \times 10^{-1}$</td>
<td>$3.3 \times 10^{-1}$</td>
<td>$2.8 \times 10^{-1}$</td>
<td>$1.5 \times 10^{-1}$</td>
<td>$2.9 \times 10^{-1}$</td>
<td>$3.8 \times 10^{-1}$</td>
<td>$3.8 \times 10^{-1}$</td>
</tr>
<tr>
<td>$1.6 \times 10^{-1}$</td>
<td>$1.2 \times 10^{0}$</td>
<td>$5.3 \times 10^{-1}$</td>
<td>$1.5 \times 10^{-1}$</td>
<td>$3.8 \times 10^{-1}$</td>
<td>$1.6 \times 10^{-1}$</td>
<td>$1.6 \times 10^{-1}$</td>
</tr>
</tbody>
</table>

Note the number of elements which are of order unity. This difference in the second scenario is as follows:

<table>
<thead>
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<th></th>
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<th>$3.2 \times 10^{-3}$</th>
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<th>$-2.2 \times 10^{-2}$</th>
<th>$5.8 \times 10^{-2}$</th>
<th>$9.9 \times 10^{-4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3.2 \times 10^{-3}$</td>
<td>$-1.9 \times 10^{-3}$</td>
<td>$2.3 \times 10^{-2}$</td>
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</tr>
<tr>
<td>$-2.2 \times 10^{-2}$</td>
<td>$2.4 \times 10^{-2}$</td>
<td>$-4.4 \times 10^{-4}$</td>
<td>$-1.3 \times 10^{-3}$</td>
<td>$-2.1 \times 10^{-3}$</td>
<td>$2.1 \times 10^{-2}$</td>
<td>$2.1 \times 10^{-2}$</td>
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<tr>
<td>$5.8 \times 10^{-2}$</td>
<td>$1.9 \times 10^{-2}$</td>
<td>$8.3 \times 10^{-3}$</td>
<td>$-2.1 \times 10^{-3}$</td>
<td>$7.5 \times 10^{-3}$</td>
<td>$1.7 \times 10^{-2}$</td>
<td>$1.7 \times 10^{-2}$</td>
</tr>
<tr>
<td>$9.9 \times 10^{-4}$</td>
<td>$3.1 \times 10^{-1}$</td>
<td>$2.2 \times 10^{-2}$</td>
<td>$2.1 \times 10^{-2}$</td>
<td>$1.7 \times 10^{-2}$</td>
<td>$1.6 \times 10^{-2}$</td>
<td>$1.6 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

The initial value of

$$
\left| \frac{P_y - \frac{\partial f}{\partial x} P_x \frac{\partial f}{\partial x}^T}{|P_y|} \right|
$$

is 0.33 for the scenario presented here whereas this quantity is 3.08 for the first scenario. Hence the error made in approximating the first step covariance by the
first order expression in terms of $P_z$ is larger for the first scenario than for the second.

Clearly the first step state covariance in the second scenario is much closer to the first order approximation. For this reason, both the coordinate transformation filter and the six state two-step filter have very similar performance. Both of the filters have mean errors close to that of the state augmented two-step estimator.

6.4 Summary of Numerical Results

The Monte Carlo simulations presented in this chapter demonstrate that the two-step filter can provide a better mean square state estimate than the IEKF and the coordinate transformation based Kalman filter under two situations. The first is when the initial error is very large as compared to the expected state uncertainty and the second is with systems that have a low dimensional observation vector and as such are dependent upon the filter combining measurements over time to make the state observable. This advantage has only really been shown to be significant for the latter situation.

The rank deficiency problem appeared to be worse for the $n = m$ two-step filter than for the state augmented two-step filter. This finding may indicate an advantage in using a state augmented filter even if, theoretically, the $n = m$ two-step filter is expected to have equivalent performance.

The importance of properly initializing the first step covariance matrix in order to realize the complete benefits of the two-step filter has been demonstrated. This is especially true for the $n = m$ situation, which may give further justification for use of a state augmented filter. On the other hand, if the first order approximation to $P_y$ is good, than the simpler coordinate transformation filter is expected to give estimation accuracy comparable to the two-step filter, and better
than that of the IEKF.
CHAPTER 7

CONCLUSION

The two-step filter can provide a better recursive nonlinear least squares estimate to be used for the relative navigation of two satellites in an elliptical orbit under some circumstances. These include large a priori state error in the initialization of the filter and use of a small dimension observation vector which thus requires the incorporation of many measurements over time to make the state observable. The simulations in this thesis have shown that the improvement of the two-step estimator over the iterated extended Kalman filter is really only justified in the latter case. Furthermore, the significance of any improvement over a coordinate transformation based extended Kalman filter will depend upon the accuracy of the first order approximation to the first step state covariance matrix. However, a number of potential problems in the application of this filter must be recognized and dealt with before it can be used in a robust and reliable manner for this, or any other similarly complex problem.

The most important difficulty which have been identified in the work in this thesis is the potential to generate a numerically rank deficient first step covariance matrix. It has also been demonstrated that the two-step filter with equal dimension first and second step states, as well as the similar coordinate transformation based EKF, can occasionally generate a divergent solution.

Future work is needed in these areas to show that these concerns are properly accounted for before this filter is used in any real time mission critical application. Possible directions of future work include techniques for selecting first step states: different formulations of the first step state and covariance time update
equations so as to avoid non-positive definite first step covariance matrices: special cases of equal dimensioned states which can be solved in closed form and hence eliminate the iteration; and automatic residual monitoring techniques which can detect the onset of filter divergence and make necessary corrections.

The elliptical orbit relative navigation problem is a good application of these techniques because the nonlinearities in the dynamics and measurements are significant. Also, the observation vector is usually small and hence the filter must be relied upon to combine many different measurements to get sufficient state observability. Furthermore, operations of satellite “clusters” could be done cheaper with on-board orbit determination capability without having to rely upon the direct intervention of a ground based tracking network. This is especially true if a reliable relative navigation solution could be obtained using only range or range-rate. In this case, the position of each satellite in the cluster could be maintained independently using very simple hardware on the secondaries and only requiring orbit determination of the primary vehicle. More robust and accurate nonlinear filtering techniques are an important technical contribution to make such operations possible.
BIBLIOGRAPHY

[Aidala and Hammel, 1983] Aidala, Vincent J. and Hammel, Sherry E.,


[Balakrishnan and Speyer, 1985] Balakrishnan, S. N., and Speyer, Jason L.,


APPENDIX A

FILTER ALGORITHMS

A.1 UD Covariance Factorization Two-Step Filter Algorithm

(Including modifications to reduce the effects of ill-conditioned first step covariance matrices)

(1) Initialization of the following:

\[ P_{x_{0}}(+), \quad P_{y_{0}}(+), \quad x_{0}(-), \quad y_{0}(-) \]  \hspace{1cm} (A.1)

(2) Propagation of the second step states and the state transition matrix

\[ \Phi_{(i,i-1)} \] by numerical integration from the \( i - 1 \text{ST} \) to the \( i \text{TH} \) time step.

The first step state is estimated from

\[ \hat{y}_{i}(-) = \hat{y}_{i-1}(+) + f(\hat{x}_{i}(-)) - f(\hat{x}_{i-1}(+)) \]  \hspace{1cm} (A.2)

The second step covariance is propagated by computing the block matrices \( Y_{x} \) and \( \tilde{D}_{x} \)

\[ Y_{x} = \begin{bmatrix} \Phi_{(i,i-1)} U_{x_{i-1}}(+) & G_{d} \end{bmatrix} \]  \hspace{1cm} (A.3)

\[ \tilde{D}_{x} = \begin{bmatrix} D_{x_{i-1}}(+) & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & Q_{d} \end{bmatrix} \]  \hspace{1cm} (A.4)

and then reducing them to \( m \) by \( m \) upper triangular and diagonal matrices, \( U_{x_{i}}(-) \) and \( D_{x_{i}}(-) \), respectively, such that

\[ U_{x_{i}}(-)D_{x_{i}}(-)D_{x_{i}}^{T} = Y_{x} \tilde{D}_{x} Y_{x}^{T} = P_{x_{i}}(-) \]  \hspace{1cm} (A.5)

This is done using the Modified Weighted Gram-Schmidt (MWGS)

Similarly, the subsequent propagation of the first step states as defined in Eq (2.53) is done by computing the matrices \( Y_y \) and \( \bar{D}_y \).

\[
Y_y = \begin{bmatrix} D_{y,-1}(+) & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & D_{z_i}(-) & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & -D_{z_{i-1}}(+) & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & \epsilon I \\
\end{bmatrix}
\]

(A.6)

\[
\bar{D}_y = \begin{bmatrix} U_{y,-1}(+) & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & D_{z_i}(-) & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & -D_{z_{i-1}}(+) & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots & \epsilon I \\
\end{bmatrix}
\]

(A.7)

The MWGS process is then used to compute \( U_{y_i}(-) \) and \( D_{y_i}(-) \) such that

\[
U_{y_i}(-)D_{y_i}(-)U_{y_i}^T(-) = Y_y \bar{D}_y Y_y^T
\]

(A.8)

(3) If necessary, perform some check on the condition of the matrix \( P_{y_i}(-) \) generated in step 2. If this indicates numerical problems, return to without updating the state.

(4) The measurement update is performed by converting the observation vector \( \tilde{z}_i \) into \( l \) uncorrelated scalar observations \( \tilde{z}^* \) with unity variance through a coordinate transformation.

\[
WW^T = R_i
\]

(A.9)

\[
\tilde{z}^* = W^{-1}\tilde{z}
\]

(A.10)

\[
H^* = W^{-1}H
\]

(A.11)

Details of the scalar measurement update are given in [Mayheek 1979]. This step will result in the a posteriori first step states \( \hat{y}_i(+) \) and the \( U'_{y_i}(+) \)
and $D_{y_i}(+)$ matrices. The inverse of $P_{y_i}^{-1}(+)$ can then be computed by the more numerically stable method

$$
P_{y_i}^{-1}(+) = U_{y_i}^{-T}(+) U_{y_i}^{-1}(+)$$

(5) The a posteriori second step states are found from the $\tilde{y}_i(+), U_{y_i}(+)$ and $D_{y_i}(+)$ by numerically minimizing the cost function

$$J(\hat{\epsilon}) = \frac{1}{2}(\tilde{y}_i(+) - f(\hat{\epsilon}))^T P_{y_i}^{-1}(+) (\tilde{y}_i(+) - f(\hat{\epsilon}))$$

The Lavenberg-Marquardt method [Press, et al., 1994] is used for this numerical minimization.

The iterations are initialized with $\tilde{\epsilon}^0 = \tilde{\epsilon}_i(-)$ and $\lambda = 0.0001$. At each $k^{TH}$ iteration, the Hessian is computed from the approximation

$$H_G \approx \frac{\partial f}{\partial x}^T |_{\hat{\epsilon}_k} P_{y_i}^{-1}(+) \frac{\partial f}{\partial x} |_{\hat{\epsilon}_k}$$

(A.14)

The linear system

$$[\alpha] \delta \hat{\epsilon} = - \frac{\partial J(\hat{\epsilon})}{\partial \hat{\epsilon}} |_{\hat{\epsilon}_k}$$

(A.15)

is solved to update the state estimate

$$\hat{\epsilon}^{k+1} = \hat{\epsilon}^k + \delta \hat{\epsilon}$$

(A.16)

in which the gradient of $J$ is

$$\frac{\partial J}{\partial \hat{\epsilon}} |_{\hat{\epsilon}_k} = -(\tilde{y}_i - f(\hat{\epsilon}_k)) P_{y_i}^{-1}(+) \frac{\partial f}{\partial x} |_{\hat{\epsilon}_k}$$

(A.17)

and $\alpha$ is computed from

$$\alpha_{i,t} = H_{G_{i,t}} (1 + \lambda)$$

(A.18)
\[ \alpha_{p,q} = H_{p,q} \quad \text{for } p \neq q \]  

(A.19)

The change in the cost function \( J(\hat{x}^{k+1}) \) determines how the next iteration is performed.

- If \( J(\hat{x}^{k+1}) \geq J(\hat{x}^k) \) then
  \[ \lambda = 10\lambda \]
  \[ k = k + 1 \]
  and execution is returned to \( \dagger \).

- If \( J(\hat{x}^{k+1}) < J(\hat{x}^k) \) then
  \[ \lambda = \lambda/10 \]
  \[ \hat{x}^{k+1} = \hat{x}^k \]
  \[ k = k + 1 \]
  and execution is returned to \( \ddagger \).

The iterations are continued until the change in each element of \( \hat{x}^{k+1} \) is less than some tolerance after a minimum number of iterations have been performed. After this indication of convergence, the \textit{a posteriori} second step states are updated by

\[ \hat{x}_1(+) = \hat{x}^k \]  

(A.20)

(6) The factors of the \textit{a posteriori} second step covariance \((U_{x_1}(+) \text{ and } D_{x_1}(+))\)

are found by computing

\[ S(+) = U_{y_1}(+) \left. \frac{\partial f}{\partial \hat{x}} \right|_{\hat{x}^k(+) \hat{x}_1(+) \hat{x}_2(+) \hat{x}_3(+) \hat{x}_4(+) \hat{x}_5(+) \hat{x}_6(+) \hat{x}_7(+) \hat{x}_8(+)} \]  

(A.21)

and using \( D_{y_1}(+) \) from (A.12). The MWGS method is then used to produce \( U_{x_1}(+) \) and \( D_{x_1}(+) \) from

\[ [U_{x_1}(+)]^{-1} [D_{x_1}(+)]^{-1} \left[ U_{x_1}^T(+) \right]^{-1} = S^T(+) D_{y_1}(+) S(+) \]  

(A.22)
A.2 UD Covariance Factorization Two-Step Filter Algorithm for \( n = m \)

This is the same as for the \( n > m \) two-step filter, except that steps (5) and (6) can be replaced by the following, which does not require iteration.

\[
\hat{x}_i(+) = f^{-1}(\hat{y}_i(+) ) 
\]  
(A.23)

Compute the factors \( Y \) and \( \hat{D} \).

\[
Y = \left. \frac{\partial f^{-1}}{\partial x} \right|_{\hat{y}_i(+) } U_{y_i}(+) 
\]  
(A.24)

\[
\hat{D} = D_{y_i}(+) 
\]  
(A.25)

Use the MWGS method to produce \( U_{x_i}(+) \) and \( D_{x_i}(+) \) from

\[
U_{x_i}(+) D_{x_i}(+) D_{x_i}^T(+) = Y_x \hat{D} Y_x^T 
\]  
(A.26)

A.3 UD Covariance Factorization Iterated Extended Kalman Filter Algorithm

(1) Initialization of the following:

\[
P_{x_0}(-), \quad \hat{x}_0(-) 
\]  
(A.27)

(2) \( ^\dagger \) Propagation of the state and the state transition matrix \( \Phi_{i,i-1} \) by numerical integration from the \( i-1^{ST} \) to the \( i^{TH} \) time step. The covariance is propagated by computing the block matrices \( Y_x \) and \( \hat{D}_x \)

\[
Y_x = \begin{bmatrix} \Phi_{i,i-1} U_{x_{i-1}}(+) & G_d \end{bmatrix} 
\]  
(A.28)

\[
\hat{D}_x = \begin{bmatrix} D_{x_{i-1}}(+) & 0 \\
\cdots & \cdots & \cdots \\
0 & Q_d \end{bmatrix} 
\]  
(A.29)

and then reducing them to a \( m \) by \( m \) upper triangular and diagonal matrices, \( U_{x_i}(-) \) and \( D_{x_i}(-) \), respectively, such that

\[
U_{x_i}(-) D_{x_i}(-) D_{x_i}^T(-) = Y_x \hat{D}_x Y_x^T = P_{x_i}(-) 
\]  
(A.30)
This is done using the Modified Weighted Gram-Schmidt (MWGS) [Maybeck 1979] method.

(3) For each iteration, the linearized measurement matrix is computed based upon the updated state estimate

\[ H = \frac{\partial h}{\partial x} \bigg|_{\tilde{x}^k} \]  \hspace{1cm} (A.31)

The observation vector \( \tilde{z}^i \) is transformed into \( l \) uncorrelated scalar observations (\( \tilde{z}^* \)) with unity variance through the coordinate transformation in (A.10). The \textit{a posteriori} first step covariance \( U_{y_i}(+) \) and \( D_{y_i}(+) \) factors and the Kalman gain, \( K \) are computed. The next iteration of the state update is then computed from

\[ \tilde{x}^{k+1} = \tilde{x}_i(-) + K \left[ z^* - h_i(\tilde{x}_i^{k-1}) + H(\tilde{x}_i(-) - \tilde{x}_i^{k-1}) \right] \]  \hspace{1cm} (A.32)

Iterations are continued until the change in the state at each update changes by less than some tolerance.

(4) Return to \( \dagger \).

A.4 UD Covariance Factorization Extended Kalman Filter Algorithm Using a Coordinate Transformation

This is the same as the \( n = m \) two-step filter except that (1) and (2) are replaced by (1) and (2) below

(1) Initialization of the following:

\[ P_{x_o}(+), \quad E_o(+) \] \hspace{1cm} (A.33)

Then \( P_{y_0}(+) \) and \( \tilde{y}_0(+) \) are computed from the approximations.

\[ \tilde{y}_0(+) = f(\tilde{x}_o(+)) \] \hspace{1cm} (A.34)

\[ P_{y_0}(+) = \frac{\partial f}{\partial x} \bigg|_{\tilde{x}_o(+)} \quad P_{z_0}(+) = \frac{\partial f}{\partial x} \bigg|_{\tilde{x}_o(+)}^T \] \hspace{1cm} (A.35)
(2) Propagation of the second step states and the state transition matrix 
\( \Phi_{(i,i-1)} \) by numerical integration from the \( i-1^{ST} \) to the \( i^{TH} \) time step.

The second step covariance is propagated as in (2) of the two-step filter.

The first step state is estimated from

\[
\hat{y}_i(-) = f(\hat{x}_i(-))
\]  \hspace{1cm} (A.36)

The propagation of the first step state covariance is done by computing the matrices \( Y_y \) and \( \hat{D}_y \).

\[
Y_y = \frac{\partial f}{\partial x}\bigg|_{\hat{x}_i(-)} U_{x_i}(-)
\]  \hspace{1cm} (A.37)

\[
\hat{D}_y = D_{x_i}(-)
\]  \hspace{1cm} (A.38)

The MWGS process is then used to compute \( U_{y_i}(-) \) and \( D_{y_i}(-) \) such that

\[
U_{y_i}(-) D_{y_i}(-) U_{y_i}^T(-) = Y_y \hat{D}_y Y_y^T
\]  \hspace{1cm} (A.39)