Reduced Cost Maneuver Design Using Surrogate Models

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Abstract

This paper uses surrogate models to reduce the computational cost associated with spacecraft maneuver design in three-body dynamical systems. Sampling-based least squares regression is used to project the system response onto a set of orthogonal bases, providing a representation of the \( \Delta V \) required for rendezvous as a reduced-order surrogate model. Models are presented for mid-field rendezvous maneuvers for spacecraft in orbits in the Earth-Moon circular restricted three-body problem, including a halo orbit about EML-2 and a distant retrograde orbit (DRO) about the Moon. In each case, the initial position of the spacecraft, the time of flight, and the separation between the chaser and the target vehicles are all considered as design inputs. The results show that sample sizes on the order of \( 10^2 \) are sufficient to produce accurate surrogates, with RMS errors reaching 0.2 m/s for the halo orbit and falling below 0.01 m/s for the DRO. The expansion coefficients solved for in the surrogates are then used to conduct a global sensitivity analysis of the \( \Delta V \) on each of the input parameters, which identifies the separation between the spacecraft as the primary contributor to the \( \Delta V \) cost. Finally, the models are demonstrated to be useful for cheap evaluation of the cost function in constrained optimization problems seeking to minimize the \( \Delta V \) required for rendezvous. These surrogate models show significant advantages for maneuver design in three-body systems, in terms of both computational cost and capabilities, over traditional Monte Carlo

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methods.

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1. Introduction

In the modeling of complex systems, a trade-off typically exists between computational expense and high-fidelity analysis capabilities. One answer that has come about in response to this trade-off is to represent the system as a reduced-order surrogate model, which minimizes the need for repeated integration and other computationally expensive techniques during analysis (Simpson et al., 2001a; Queipo et al., 2005; Jin et al., 2001). Surrogate models have been proven to be an effective means of representing and estimating complex systems involving both large numbers of design variables and inherent uncertainty. Such models have been used in a broad range of aerospace disciplines, including structural design (Barthelemy and Haftka, 1993), rocket engine component design (Farmer et al., 2000; Papila et al., 2002; Vaidyanathan et al., 2000), cavitating flows (Goel et al., 2008), helicopter rotor blade vibrations (Glaz et al., 2009), and missile performance (Riddle et al., 2009). More recently, surrogate modeling has been applied to the field of astrodynamics, including interplanetary transfers and satellite constellation design (Gano et al., 2006), low-thrust trajectory optimization (Peng et al., 2013), and orbit uncertainty propagation (Jones and Doostan, 2013; Jones et al., 2013).

This paper applies surrogate models to the design of impulsive spacecraft rendezvous maneuvers, taking advantage of the models’ ability to represent complex trade spaces for rapid analysis and optimization. The paper explores the ability of surrogate models to aid in the maneuver design process in the context of spacecraft rendezvous in the Earth-Moon circular restricted three-body system, including spacecraft in halo orbits about the Earth-Moon L₂ libration point (EML-2) and in distant retrograde orbits (DRO) about the Moon. The complex nature of the three-body system precludes the use of analytical approximations, necessitating numerical simulations and creating an opportunity for reduced computational cost using the surrogate models. Using these models as efficient design tools can enable significant time savings when performing, for example, trajectory design over long propagation times, trajectory optimization using massive grid searches, or trade
space exploration for exercises in rapid mission design. The development of rapid trajectory design and optimization tools such as these were identified by Quadrelli et al. (2013), in particular, as a priority for enabling future planetary science missions.

Many types of surrogate models have been developed for design, optimization, and analysis, including, for example, polynomial regression/interpolation (Draper and Smith, 1998; Boyd, 2001), Kriging (Matheron, 1963; Simpson et al., 2001b), and radial basis functions (see Buhmann (2000) for a survey of the radial basis methods). In each case, the surrogate model seeks to use a relatively small number of sample points to generate a global function capable of approximating the system response for a reduced computational cost compared to traditional Monte Carlo methods. In addition to reducing the computational cost, surrogate models provide the ability to perform sensitivity analysis of the system on the input parameters under consideration.

Regression models and radial basis functions both fall under the category of spectral methods, which develop the global surrogate by projecting the system response onto a collection of basis functions. In regression modeling, bases are typically selected to be orthogonal with respect to some weight function (Boyd, 2001), while radial bases are radially symmetric functions based on Euclidean distance (Buhmann, 2000). In the Kriging method, on the other hand, the global function is comprised of a linear system trend and fluctuations about that trend (Simpson et al., 2001b; Queipo et al., 2005).

Regardless of the specific model chosen, the model development process includes three steps common to all surrogate models: design of experiments (DOE), estimation of the surrogate, and model validation (Queipo et al., 2005; Koziel et al., 2011). The DOE involves the selection of the sample points used to generate realizations of the solution of interest, also known as the training data. In model estimation, the true system response is solved for at each of the sample points selected in the DOE, and these solutions are then used to estimate the parameters that describe the surrogate. Finally, the resulting model is validated by assessing its predictive capabilities and performing error analysis. Although it is conceptually convenient to describe the model development as a sequential process, it should be noted that the process is frequently non-linear and iterative in practice.

This paper takes a comprehensive approach to the model development process for modeling the impulsive $\Delta V$ required for spacecraft rendezvous in the circular restricted three-body problem (CRTBP) as a function of deterministic design inputs. In the DOE, a novel sampling method developed in
Hampton and Doostan (2014, 2015) is introduced, which is shown to lead to more rapid convergence of the model solution. In model estimation, least squares regression is selected as the model of choice due to the adaptability of the basis functions to best suit the system under consideration. The functions can be chosen according to the boundary conditions associated with deterministic design inputs, as is done here, or alternatively, if the surrogate is mapping uncertainties in stochastic input parameters through a dynamical system, the bases can be selected according to the probability distribution of those stochastic inputs (Wiener, 1938; Ghanem and Spanos, 1991). This method, known as polynomial chaos (PC) expansions, is very useful for uncertainty quantification and propagation (Xiu, 2010; Le Maître and Knio, 2010). It has the potential to be combined with non-stochastic regression models for the purpose of optimization under uncertainty (OUU) (Eldred, 2009, 2011), in which the basis functions would take the form of the product of the bases generated for the deterministic and stochastic dimensions. Thus, another benefit of using regression modeling for this work is to serve as the foundation for future work in developing an accurate and cost-effective method for OUU as applied to spacecraft maneuver design. Finally, in model validation, this paper considers two separate validation data sets, one useful for providing a comprehensive analysis of the modeling errors and the other more conducive to rapid, autonomous model development.

Section 2 of the paper provides a review of the dynamics of the CRTBP and defines the orbits to be considered for the rendezvous problem. Section 3 includes the details of model development for least squares regression, including the design of experiments, model estimation, and model validation, in addition to introducing a technique for global sensitivity analysis enabled by the surrogate. The results of the models built for rendezvous in both a halo orbit and a DRO are presented in Section 4, and finally, Section 5 applies the results from Section 4 to a constrained optimization of the rendezvous maneuver.

2. Problem Setup

This paper is concerned with spacecraft in the Earth-Moon three-body system and particularly considers the dynamics defined by the CRTBP. Thus,
the system dynamics of interest for the model are, in dimensionless form,

\[
\ddot{x} - 2\dot{y} - x = -\frac{(1 - \mu)(x - \mu)}{r_1^3} - \frac{\mu(x + 1 - \mu)}{r_2^3}
\]

\[
\ddot{y} + 2\dot{x} - y = -\frac{(1 - \mu)y}{r_1^3} - \frac{\mu y}{r_2^3}
\]

\[
\ddot{z} = -\frac{(1 - \mu)z}{r_1^3} - \frac{\mu z}{r_2^3},
\]

where

\[
r_1 = \sqrt{(x + \mu)^2 + y^2 + z^2}, \quad r_2 = \sqrt{(x + \mu - 1)^2 + y^2 + z^2}
\]

and \(\mu\) is the mass ratio of the Earth-Moon system,

\[
\mu = \frac{m_M}{m_E + m_M}.
\]

A surrogate model is first built for spacecraft orbiting in a 10,000 km \(z\)-amplitude halo orbit about the EML-2 libration point, as illustrated in Fig. 1. Specifically, the surrogate is used to model the mid-field rendezvous phase, in which a two-burn sequence carries the chaser vehicle from orbit insertion to the start of its proximity operations with the target (Hinkel et al., 2014). The design inputs include the initial position of the chaser vehicle, as measured by the \(\tau\)-value (see Fig. 1a); the separation between the chaser and target vehicles \(\Delta \tau\); and the time of flight \(t\). Table 1 presents the range of values considered for these design parameters. The quantity of interest to be modeled is the \(\Delta V\) required for rendezvous, including the magnitude of the total \(\Delta V\) and the \(x\)-, \(y\)-, and \(z\)-components of the initial and final burns. Using the three design parameters, the deterministic solutions for the required \(\Delta V\) are computed with a single-shooting differential corrector. Figure 2 shows the magnitude of the actual total required \(\Delta V\) for rendezvous in the halo orbit.

Once the surrogate framework is developed and refined for the halo orbit, it is next applied to mid-field rendezvous of spacecraft in a distant retrograde orbit (DRO) about the Moon in order to assess the model’s capabilities as a black box development tool for three-body dynamics that requires minimal additional processing.

The orbit selected is a 70,000 km DRO, as shown in Fig. 3, of particular interest to NASA for its Asteroid Redirect Mission (ARM), in which the
crewed Orion vehicle will rendezvous with an asteroid and the Asteroid Redirect Vehicle (ARV) already in orbit in the DRO. This case has previously been studied by Hinkel et al. (2014) using a Monte Carlo approach, and this paper aims to show that the use of a surrogate model can provide additional information and analysis capabilities over the Monte Carlo at a considerably reduced computational cost.

In Hinkel et al. (2014), the \( \Delta V \) cost of rendezvous is determined for spacecraft separated by \( 0.01 - 0.22 \)° for a single position \( \tau \), with a time of flight of 6, 12, or 24 hours. This paper first replicates the previous study, in which only \( \Delta \tau \) and \( t \) are included as design dimensions, and then extends the study to include the initial position of the spacecraft as a third design dimension. The design parameters for each of these cases are also provided in Table 1. The true magnitude of the \( \Delta V \) required for rendezvous in the DRO is given in Fig. 4, which illustrates that \( \tau \) can have a non-negligible effect on \( \Delta V \) that

\[ \text{Table 1: Rendezvous design space} \]

<table>
<thead>
<tr>
<th>Range</th>
<th>Halo</th>
<th>DRO (( d = 2 ))</th>
<th>DRO (( d = 3 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial position, ( \tau ) (°)</td>
<td>[0, 360]</td>
<td>0</td>
<td>[0, 360]</td>
</tr>
<tr>
<td>Time, ( t ) (days)</td>
<td>[0.25, 1]</td>
<td>[0.25, 1]</td>
<td>[0.25, 1]</td>
</tr>
<tr>
<td>Initial separation, ( \Delta \tau ) (°)</td>
<td>[0.1, 5]</td>
<td>[0.01, 0.25]</td>
<td>[0.01, 0.25]</td>
</tr>
</tbody>
</table>
Figure 2: Total required $\Delta V$ for spacecraft rendezvous in a 10,000 km z-amplitude halo orbit about EML-2

Figure 3: 70,000 km distant retrograde orbit about the Moon

is not accounted for in Hinkel et al. (2014). While the shape of the function is very similar to that seen for the halo orbit, the case of the DRO results in lower required $\Delta V$ due to the smaller values of $\Delta \tau$ being used.
\( \Delta \tau = 0.15^\circ \)  

\( t = 0.5 \text{ days} \)

Figure 4: Total required \( \Delta V \) for spacecraft rendezvous in a 70,000 km DRO

3. Surrogate Model

As a spectral method, regression modeling expresses the functional dependence of a system response on the \( d \)-dimensional system inputs as a linear combination of a set of basis functions \( \psi_i \), resulting in the series expansion

\[
u(s) = \sum_{i=1}^{\infty} c_i \psi_i(s),
\]

where \( \psi_i \) are functions of the system inputs \( s = [s_1, s_2, \ldots, s_d] \in \Gamma^d = [-1, 1]^d \), and \( c_i \) are the corresponding coefficients. This work considers basis functions \( \psi_i \) that are orthonormal, i.e.,

\[
\int_{\Gamma^d} \psi_i(s)\psi_j(s)ds = \delta_{ij},
\]

where \( \delta_{ij} \) is the Kronecker delta. In practice, under some regularity conditions, \( u(s) \) can be approximated using (possibly small) \( P \) expansion terms, such that

\[
u(s) \approx u_P(s) = \sum_{i=1}^{P} c_i \psi_i(s),\]

\( \times \)
for an appropriate ordering of the basis functions $\psi_i$. Thus, the modeling error $\varepsilon(s)$ associated with the surrogate is the truncation error given by

$$
\varepsilon(s) = \sum_{i=P+1}^{\infty} c_i \psi_i(s).
$$

(4)

The exact solution of the coefficients corresponding to the expansions in Eqs. (2) and (3) is the projection of the solution $u(s)$ onto the bases $\psi_i$,

$$
c_i = \langle u(s) \psi_i(s) \rangle = \int_{\Gamma^d} u(s) \psi_i(s) ds,
$$

(5)

where it is assumed that

$$
\int_{\Gamma^d} u^2(s) ds < \infty.
$$

When Eq. (5) is used to solve for the truncated model of Eq. (3), increasing the number of terms $P$ results in mean-squares convergence of the estimate $u_P(s)$ to the true response $u(s)$, i.e.,

$$
\langle (u(s) - u_P(s))^2 \rangle \rightarrow 0 \quad \text{as} \quad P \rightarrow \infty.
$$

(6)

In multidimensional systems, the $\psi_i$ of Eq. (3) are taken to be the tensor product of the univariate basis functions $\psi_{\alpha_k}(s_k)$ in each design dimension $s_k$, $k = 1, \ldots, d$, such that

$$
\psi_{\alpha}(s) = \psi_{\alpha_1}(s_1) \times \cdots \times \psi_{\alpha_d}(s_d).
$$

(7)

For polynomial bases, $\alpha = (\alpha_1, \ldots, \alpha_d)$ denotes a multi-index defined for total order bases, for instance, by

$$
\alpha \in \Lambda_{p,d} = \{ (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d : \|\alpha\|_1 \leq p \},
$$

for which $\alpha_k$ denotes the degree of the univariate polynomial function $\psi_{\alpha_k}(s_k)$, $p$ is the total degree of the expansion, and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ is the set of natural numbers. The tensor-product basis construction in Eq. (7) allows for the basis functions $\psi_{\alpha_k}$ in each dimension $s_k$ to be chosen independently (Soize and Ghanem, 2004). Eq. (3) may then be rewritten as

$$
u_P(s) = \sum_{\alpha \in \Lambda_{p,d}} c_\alpha \psi_{\alpha}(s).
$$

(8)
The number of terms $P$ in the resulting polynomial expansion is dependent on both $p$ and the number of input dimensions $d$ and is given by

$$P = |\Lambda_{p,d}| = \frac{(p + d)!}{p!d!}.$$ 

The enrichment of the approximation given by $u_P$ in Eq. (8) is therefore achieved by increasing the total order of expansion $p$, which results in retaining a larger number of expansion terms $P$.

Many types of functions are available for use as the bases in spectral expansions. In regression modeling, $\psi_i$ may be chosen to be orthogonal bases dependent on the support of the design parameters (Boyd, 2001). Of particular interest here are problems with periodic boundary conditions, e.g. the initial $\tau$ of the chaser spacecraft in the rendezvous problem, and those with non-periodic boundary conditions defined over a finite interval, e.g. the $\Delta \tau$ and $t$ dimensions in the same problem. Non-periodic systems with a finite domain may be represented using either Chebyshev or Legendre polynomials. Legendre polynomials have been selected for this work, such that the basis functions take the form

$$\psi_i(s) = L_i(s),$$

where $L_i(s)$ is the Legendre polynomial of degree $i$.

Periodic systems have been shown to be best modeled using a Fourier expansion, for which the non-constant basis functions are

$$\psi_i(s) \in \{\cos (i\pi s), \sin (i\pi s)\}.$$ 

Because the Fourier expansion contains both sine and cosine terms, however, the use of these bases in the expansion of Eq. (3) results in a number of terms equal to

$$P' = 2^{d_F} \cdot P - d_F,$$

where $d_F$ is the number of design dimensions along which the solution is approximated with a Fourier expansion. Here, both the Fourier and Legendre bases are defined for $s \in [-1, 1]$.

Exponential bases of the form

$$\psi_i(s) = e^{-is}$$

were also considered for the time of flight dimension of the rendezvous problem, as the required $\Delta V$ is expected to grow exponentially as $t \to 0$. However,
no improvement was seen over the use of the Legendre bases, likely because the non-orthogonality of the exponential bases can lead to correlation between the regression coefficients, making the exponential bases non-optimal (Montgomery and Peck, 1992). Thus, the exponential model has been omitted from the results contained in Section 4.

3.1. Design of Experiments

The primary concern of the design of experiments (DOE) is the sampling method used to select the training data, i.e., realizations of \( s \) and \( u(s) \), with which to generate the surrogate model. Monte Carlo sampling provides a basic approach, in which the input data points are randomly selected over the design space \( \Gamma^d \). For both Legendre and Fourier bases, this corresponds to a uniform sampling of \( \Gamma^d \).

Techniques such as Latin hypercube sampling (LHS) and orthogonal arrays (OA), which are modifications on traditional Monte Carlo sampling, have been developed to reduce model bias by attempting to provide a better distribution of samples and prevent clustering of data in the training set (Queipo et al., 2005). However, one major advantage of Monte Carlo sampling is that it allows for the reuse of data points when the size of the training set is adjusted. The structured approach to sampling in LHS and OA can preclude this recycling of data and require that an entirely new set of data be generated when it is determined that a larger sample size is needed to improve model accuracy.

Another common approach to sampling uses a deterministic set of points to generate the model rather than random sampling. One such technique is known as pseudospectral collocation, in which the training points are chosen to be quadrature nodes, such as Gaussian quadratures. In the multidimensional case, the sample data can be taken as the tensor product of the quadrature nodes in each of the \( d \) input dimensions, but this approach is subject to the curse of dimensionality, which states that the number of total function evaluations \( M \) grows exponentially with respect to the dimensionality \( d \). To limit the growth in the number of nodes required to solve the model in high dimensional problems, sparse grids based on, for example, Smolyak’s algorithm can be used, which neglect higher-order interactions among dimensions (Xiu, 2010). Finally, other deterministic techniques such as A-, D-, and E-optimal seek to improve the stability of the linear system of Eq. (12) (discussed in the following section), from which the expansion coefficients are computed (Santner et al., 2003; Pukelsheim, 1993).
3.2. Estimation of the Model

The solution of the integral in Eq. (5) to generate the surrogate model is generally not straight-forward, therefore methods have been developed to determine a best estimate of the coefficients \( c_i \) using the system response at the training points selected in the DOE. Sampling-based methods achieve this estimation by relying only on the system inputs and outputs, thus treating the system solver as a black box. Two commonly-used sampling-based methods for generating a surrogate model are least squares regression and pseudospectral collocation. For the method of least squares, which is used here, \( u(s) \) is evaluated at random sample points \( \{s^{(i)}\} \) and Eq. (3) is written in matrix form as

\[
\mathbf{u} \approx \Psi \mathbf{c},
\]

where \( \mathbf{u} = (u(s^{(1)}), \ldots, u(s^{(M)}))^T \) is the vector of realizations of the system response at each of the \( M \) sample points \( s^{(i)} \), \( \mathbf{c} = (c_1, \ldots, c_P)^T \) is the vector of expansion coefficients, and \( \Psi \) is an \( M \times P \) matrix containing evaluations of the basis functions for each term of the expansion at each sample. Using the notation \( \{\psi_j\}, j = 1, \ldots, P \), with a one-to-one correlation between \( \{\psi_j\} \) and \( \{\psi_\alpha\} \), the \((i,j)\)-th element of \( \Psi \) is defined as

\[
\Psi_{(i,j)} = \psi_j(s^{(i)}).
\]

The least squares estimation, then, seeks to solve for \( \mathbf{c} \) such that the sum of the squares of differences between the model prediction \( u_P(s) \) and the system response \( u(s) \) is minimized at the \( M \) sample points, or

\[
\hat{\mathbf{c}} = \arg \min_{\mathbf{c}} \| \mathbf{u} - \Psi \mathbf{c} \|_2^2.
\]

This corresponds to the solution of the normal equation

\[
(\Psi^T \Psi) \hat{\mathbf{c}} = \Psi^T \mathbf{u}.
\]

For a discussion on the analysis of least squares regression and the required number of samples \( M \) for successful solution recovery, the interested reader may refer to Cohen et al. (2013) and Hampton and Doostan (2014). Related to least squares regression, recent techniques based on compressive sampling exploit the sparsity in \( \mathbf{c} \) to reduce \( M \), and hence the computation cost, required for a successful solution recovery (Doostan and Owhadi, 2011; Hampton and Doostan, 2014).
While the construction of the surrogates was described here in the context of a scalar quantity of interest (QOI), the technique can also be applied to vector-valued QOIs with no required increase in $M$. For $N$ QOIs, the vector $\mathbf{u}$ in Eq. (9) becomes an $M \times N$ matrix, and, similarly, the estimated matrix $\mathbf{c}$ is of dimension $P \times N$. Additional information on surrogate solutions for vector-valued QOIs can be found in Jones and Doostan (2013).

3.3. Validation

The third step in the model development process involves assessing the ability of the model to accurately predict the true system response. Inherent in this step is the selection of the validation data set, which is ideally independent of the data used in the construction of the surrogate model. Model validation is useful in gaining a more complete understanding of any modeling errors and, if multiple surrogates are being considered, in the selection of the surrogate best suited for the system.

To enable more robust analysis during the initial evaluation of the suitability of least squares regression in modeling impulsive maneuvers in the CRTBP, the models developed here are first compared to a very large validation set consisting of a grid of data points uniformly distributed along each input dimension, for a total number of points on the order of $10^5$ for the case of the halo orbit and $10^4$ for the DRO. The grid size in each dimension, as well as total number of resulting validation points, are provided in Table 2. Using these points, the models are evaluated based on the RMS validation errors, calculated as

$$\varepsilon_2 = \sqrt{\frac{1}{M_v} \sum_{i=1}^{M_v} (u_P(s^{(i)}) - u(s^{(i)}))^2},$$

(13)

where $M_v$ is the number of validation points, $u_P(s^{(i)})$ is the model estimate at these points, and $u(s^{(i)})$ is the true value at each point.

This approach to model validation, though, is clearly not practical for general applications and rapid model development. Other methods have been designed for model validation which impose less computational cost while still providing a thorough and unbiased assessment of the model errors, e.g. $k$-fold cross-validation and bootstrapping, among others (Geisser, 1975; Queipo et al., 2005). In particular, $k$-fold cross-validation, applied in this study, has the advantage of allowing all available sample data to be used for
Table 2: Validation data sets

<table>
<thead>
<tr>
<th></th>
<th>Step Size</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Initial</td>
<td>Time,</td>
<td>Initial</td>
</tr>
<tr>
<td></td>
<td>Position,</td>
<td>t (days)</td>
<td>Separation,</td>
</tr>
<tr>
<td>Halo Orbit</td>
<td>2</td>
<td>0.05</td>
<td>0.1</td>
</tr>
<tr>
<td>DRO</td>
<td>2</td>
<td>0.05</td>
<td>0.01</td>
</tr>
</tbody>
</table>

both constructing and validating the surrogate model. The process for $k$-fold validation is as follows:

1. Divide the training data set of size $M$ into $k$ subsets of approximately equal size.
2. Construct $k$ surrogates, each omitting one subset of data to be used as the validation set.
3. Compute the model error for each estimated surrogate against its corresponding validation set.
4. Average the model error from each of the $k$ surrogates to obtain the generalization error estimate.
5. If the generalization error estimate falls within accuracy requirements, generate a final surrogate using all $M$ data points.

3.4. Analysis of Variance

Once built, surrogates can, among other uses, be utilized to conduct analysis of variance (ANOVA) for the solution of interest. ANOVA techniques are methods which provide a measure of the global sensitivity of the QOI of a system to variations in each of the system inputs. One commonly used measure, the Sobol’ sensitivity indices (Sobol’, 1993), was developed in the context of stochastic systems and represents sensitivities as a ratio of the variance in the QOI due to stochastic dimension $i$ to the total variance,

$$S_i = \frac{D_i}{D}.$$  \hspace{1cm} (14)

Although these indices can be calculated via Monte Carlo simulation, a method introduced in Sudret (2008) makes use of polynomial chaos surrogates to enable an analytic computation of the indices using the expansion coefficients.
While design surrogates are not stochastic and therefore the approach in Sudret (2008) is not directly transferable to the models developed here, a related set of indices is proposed in which the indices provide a relative measure of the dependence of the system response on input \( i \) and are defined as

\[
S_i = \frac{C_i}{C},
\]

where

\[
C_i = \sum_{\alpha_i \in \Lambda_{p,d}} c_{\alpha_i}^2.
\]

\( \alpha_i \) is the multi-index in \( \Lambda_{p,d} \) for which the \( i \)th entry is non-zero, and

\[
C = \sum_{i=1}^{d} C_i.
\]

These sensitivity indices provide two specific advantages for sensitivity analysis. First, the indices can be calculated analytically with the expansion coefficients rather than empirically using a Monte Carlo method. Second, while many approaches to sensitivity analysis, such as function derivatives, provide information on local sensitivities, the indices of Eq. (15) can instead be used for global analysis. In effect, \( \{S_i\}_{i=1}^{d} \) represent the overall contribution of each dimension to the system as a whole. In the context of spacecraft rendezvous, such an analysis provides useful information regarding which design inputs contribute most to the required \( \Delta V \) and therefore which are the most important considerations for minimizing fuel costs during mission design.

4. Model Results

In this section, numerical simulations are used to build a surrogate of the \( \Delta V \) (including the \( x \)-, \( y \)-, and \( z \)-components of the initial and final burns, as well as the total \( \Delta V \)) required for spacecraft rendezvous and to assess the model’s ability to accurately represent rendezvous in the three-body system. The full validation sets are used to evaluate the model performance in both the halo orbit and the DRO, including the magnitude and behavior of modeling errors. \( k \)-fold validation is then used to test the application of an algorithm for adaptive, autonomous model development.
To enable autonomous development, the \( k \)-fold algorithm begins with a first order expansion, generates a model, and calculates the \( k \)-fold RMS error. If the \( k \)-fold error does not meet accuracy requirements, the number of samples is increased incrementally, and a new model is generated and assessed for each new sample size. If the sample size reaches the threshold

\[
M \geq \gamma \cdot P_{\text{current}},
\]

where

\[
P_{\text{current}} = \frac{(p_{\text{current}} + d)!}{p_{\text{current}}!d!}
\]

and \( \gamma \) is a design parameter, the expansion order of the model is also increased, so that

\[
p_{\text{new}} = p_{\text{current}} + 1.
\]

The process is then repeated until the resulting modeling error meets the specified accuracy requirements. This adaptive method allows for the generation of models with no \textit{a priori} knowledge on either the required order of expansion or the sample size necessary for convergence.

4.1. Halo Orbit

A surrogate is first built for the case of rendezvous in the 10,000 km halo orbit using Fourier bases in the \( \tau \) dimension and Legendre bases in the \( t \) and \( \Delta \tau \) dimensions. Figure 5 shows the RMS validation error of the model for the total \( \Delta V \) as a function of the expansion order and sample size, averaged over 100 independent data sets for each sample size. It is readily evident that, as the order of the expansion increases, the approximation error decreases, until it reaches an error floor with a 9th order expansion. The larger errors associated with the lower order expansions correspond to the truncation error defined in Eq. (4). As the order of the expansion approaches infinity, or, in the case of lower order dynamic systems, as the expansion order approaches the true system order, the truncation error should approach zero.

The error floor, which levels off at about 0.18 m/s for large sample sizes, has three potential sources. First, because the truth data used to initialize the model is generated by numerical simulations, any round-off errors in the simulation will translate into errors in the surrogate. The accuracy levels of the single-shooter being used, though, would suggest that numerical errors contribute very little, if any, to the error floor seen here. The more likely source of the error stems from the rapid decay in the true \( \Delta V \) along the time
dimension $t$ as evidenced in Fig. 2. Generalized spectral expansions based on global smooth polynomials such as Legendre polynomials are recognized in Le Maitre et al. (2004) to face difficulties in accurately modeling a steep dependence of the QOI on an input parameter. Finally, because of this behavior in the time dimension, there simply may not exist a more accurate, low order representation of the system; rather, a very high order expansion may be required to completely capture the system response.

Despite this floor, the modeling errors drop below 0.25 m/s using on the order of only $10^2$ training points. For example, a 9th order expansion generated with 700 samples produces a validation error of 0.226 m/s. This expansion order results in $P = 385$ terms, and the corresponding coefficients can be found in Fig. 6. The drop in the coefficients of almost four orders of magnitude indicates a relatively fast convergence in the expansion. Figure 7a shows the model’s predictions for the required $\Delta V$ as a function of the initial position and time of flight for $\Delta \tau = 3^\circ$, and Fig. 7b shows the associated model errors. These errors are primarily concentrated at low values of $t$. Additionally, the errors appear to have a dominating periodic structure in the $\tau$ dimension. While the magnitude of this periodic error can be decreased by increasing the expansion order, the ability comes at the cost of significantly increasing the sample size.
4.2. Chebyshev Sampling

Here, an asymptotic sampling measure is considered as an alternative to the standard orthogonal sampling measures generally associated with orthogonal polynomials. While the Legendre polynomials naturally associate with a uniform random variable in each coordinate, the Chebyshev distribution is well suited for high order polynomial approximations (Hampton and Doostan, 2015) such as that seen in the previous section. A Chebyshev
sample is a realization of a random variable drawn from the distribution

\[ f_c(s) = \frac{1}{\pi \sqrt{1-s^2}}, \quad (20) \]

for \( s \in [-1, 1] \). If \( U \) is a random variable uniformly distributed on \([0, 1]\), \( \cos(\pi U) \) follows the Chebyshev distribution, giving a method for sampling from this distribution. In the same manner as for uniform random sampling, each coordinate is drawn independently. As this distribution differs from the orthogonality measures of the chosen polynomials, samples taken in this way require an adjustment to the computation in Eq. (12) to give a desired least squares solution, as explained in detail in Hampton and Doostan (2015). Specifically, if \( s_k^{(i)} \) denotes the realization of a sample in the \( k \)th coordinate of a sample \( s \), then let

\[ w(s^{(i)}) = \prod_{k=1}^{d} \left( 1 - (s_k^{(i)})^2 \right)^{1/4}. \quad (21) \]

For each realized \( s^{(i)} \), there is a corresponding row of \( \Psi \) and a corresponding entry of \( u \). For the computation in Eq. (12), both of these are multiplied by \( w(s^{(i)}) \). That is, Eq. (12) is instead computed by

\[ (\Psi^T W^2 \Psi) \hat{c} = \Psi^T W u, \quad (22) \]

where \( W \) is a positive diagonal matrix with entries \( w(s^{(i)}) \).

A comparison of Figs. 5 and 8 indicates that the use of Chebyshev, rather than uniform, random sampling in the \( t \) and \( \Delta \tau \) dimensions leads to more rapid convergence of the model errors, particularly for higher order expansions. Specifically, in the 9th order expansion considered previously, the substitution of Chebyshev sampling results in a slightly smaller RMS validation error (0.207 m/s) using only 500 samples, almost 30% fewer than when uniform sampling was used in all three dimensions. Even more significant than the convergence rate of the validation errors is the variance of the error over the 100 data sets. Figure 9 shows that the variance in the RMS is reduced by almost an order of magnitude when Chebyshev sampling is used.

Chebyshev sampling is therefore taken to be the superior sampling method associated with Legendre bases for the rendezvous application considered here. The coefficients for a 9th order expansion from 500 training samples, shown in Fig. 10, are very similar to those in Fig. 6 and reach the same level
of convergence. Figure 11b shows the model errors at $\Delta \tau = 3^\circ$, which are indeed smaller than those associated with uniform sampling in Fig. 7b above.

One limitation still present in the model is its difficulty in accurately predicting system behavior at the boundaries of the design space. In Fig. 11b, for example, the largest errors are concentrated near $t = 0.25$ days. Similarly, comparing Fig. 11 to Fig. 12, the model is clearly better able to capture the system response at a separation of $\Delta \tau = 3^\circ$ than at the boundary value of
Figure 10: Coefficients for a 9th order polynomial expansion, normalized to the first term coefficient, as constructed from 500 Monte Carlo samples from a uniform distribution in the $\tau$ dimension and from a Chebyshev distribution in the $t$ and $\Delta\tau$ dimensions for rendezvous in a halo orbit.

Figure 11: Model results of a 9th order polynomial expansion using 500 Monte Carlo samples from a uniform distribution in the $\tau$ dimension and from a Chebyshev distribution in the $t$ and $\Delta\tau$ dimensions for rendezvous in a halo orbit at $\Delta\tau = 3^\circ$.

Although the magnitude of the errors at $\Delta\tau = 0.1^\circ$ is generally small (see Fig. 12b), they may affect performance in applications such as optimization and should therefore be noted. Finally, Fig. 13 shows the component-wise errors for the initial and final burns. The errors, which are smaller than those corresponding to the magnitude of the total $\Delta V$, are predominantly concentrated in the $y$-direction and are largest for short transfer times.
As noted, however, the full data set used for the validation of the previous two models is not generally available when developing the surrogate. Therefore, with the alternate sampling method in place for the DOE, a new
model is developed for rendezvous in the halo orbit using the autonomous model generator and $k$-fold validation scheme outlined above, with $k = 10$ candidate surrogates. Figure 14 shows the convergence of the $k$-fold RMS error as a function of sample size. Increases in the expansion order are clearly evident in the step decreases in the RMS error, while increases in the sample size within a given order generally do not improve the model accuracy. Additionally, the $k$-fold errors appear to reach smaller values than the errors seen when using the full validation set, likely due to the spatial concentration of the larger modeling errors which are thus less likely to be represented in the smaller $k$-fold set compared to the full set. The $k$-fold RMS accuracy requirement, then, is set to 0.05 m/s, and the model generator converges on a solution using a 9th order expansion with $M = 620$ training samples; when compared to the full validation set, the model results in an RMS error of 0.1957 m/s. Figure 15 contains the model and its errors at $\Delta \tau = 3^\circ$, which are very similar to the results obtained from the fixed order model.

![Figure 14: $k$-fold RMS errors with $k = 10$, using Monte Carlo sampling from a uniform distribution in the $\tau$ dimension and from a Chebyshev distribution in the $t$ and $\Delta \tau$ dimensions for rendezvous in a halo orbit](image)

4.3. DRO

The model framework developed for rendezvous in the halo orbit is next applied to the 70,000 km DRO considered in Hinkel et al. (2014). In this NASA study, a Monte Carlo simulation is used to conduct an analysis of the total $\Delta V$ cost of mid-field rendezvous for the Asteroid Redirect Mission (ARM), in which the Orion vehicle uses a two-burn sequence to rendezvous
with the Asteroid Redirect Vehicle (ARV). The study considers spacecraft separated by approximately $0.01 - 0.22^\circ$ from a single initial position, with discrete rendezvous times of 6, 12, and 24 hours. Solutions are presented for approximately 100 isolated points for which the Orion vehicle is initially trailing the ARV and about the same number of points for which the Orion is ahead of the target.

Figure 16 shows that using the models developed here for the case considering only the $t$ and $\Delta \tau$ design dimensions with Legendre bases in each, very accurate surrogates can be built with fewer training samples than the ~100 points examined in Hinkel et al. (2014) for the case of the chaser trailing the target. In fact, an 8th order model achieves an RMS validation accuracy of $< 10^{-2}$ using only 60 samples, resulting in a full dynamical model for only about half the cost of the limited Monte Carlo analysis. In particular, the model reveals variations in the $\tau$ dimension which are missed in the previous study.

Figure 17 presents the validation errors with $\tau$ included as a third design dimension for the DRO, using Fourier bases in that dimension. While the model is still able to achieve accuracies comparable to the two-dimensional case, it requires on the order of $10^2$ training samples to do so. The errors are much smaller than those seen in the case of the halo orbit, and the error floor that existed for the halo orbit is not present. Instead, the RMS error continually approaches 0 with increasing expansion order and sample size, a
result of the smaller range of $\Delta \tau$ values under consideration.

From Fig. 17, a 9th order expansion generated from 500 training samples with an associated RMS error of $2.369 \times 10^{-3}$ m/s is selected for further analysis. Figures 18 and 19 show the resulting model and its errors. Again, the smaller range used in $\Delta \tau$ greatly improves the performance of the model, as seen in the shape of the predicted $\Delta V$ for the smallest separation angle.
Finally, with an accuracy requirement of 0.001 m/s, the k-fold-based generator once again converges on a 9th order expansion using 620 training samples. The model results for $\Delta \tau = 0.15^\circ$ can be found in Fig. 21. This last model is used to calculate the sensitivity indices of Eq. (15) for the design parameters in the DRO, and the results are shown in Table 3. From these indices, the separation between the spacecraft has the largest impact on the $\Delta V$ cost of rendezvous, with the time of flight coming in second. Not
surprisingly, the effect of the initial position within the orbit is, relatively, much smaller than that of the other two dimensions.

Figure 20: $k$-fold RMS errors with $k = 10$, using Monte Carlo sampling from a uniform distribution in the $\tau$ dimension and from a Chebyshev distribution in the $t$ and $\Delta \tau$ dimensions for rendezvous in a DRO.

Figure 21: Model results after $k$-fold convergence with $k = 10$, using Monte Carlo samples from a uniform distribution in the $\tau$ dimension and from a Chebyshev distribution in the $t$ and $\Delta \tau$ dimensions for rendezvous in a DRO at $\Delta \tau = 0.15^\circ$. 

(a) Predicted $\Delta V$  
(b) Model errors
5. Maneuver Optimization

By representing the total required $\Delta V$ in the form of an explicit function of the design inputs, the surrogates developed in the previous section are ideally suited for cheap evaluation of the cost function in constrained optimization problems seeking to minimize the $\Delta V$ for spacecraft rendezvous. The constrained minimization method used here is based on an interior point algorithm, which relies on determining a sequence of solutions to approximate minimization problems (see, for example, Byrd et al. (1999)). The interior point algorithm is implemented using Matlab’s `fmincon` optimizer. Figure 22 shows the results of the optimizer applied to the problem of minimizing the total $\Delta V$ for rendezvous in the DRO over all three design inputs, $\tau$, $t$, and $\Delta \tau$. Because `fmincon` is a local optimizer, two different results corresponding to the two local minima seen in Fig. 22 can be reached depending on the initial guess, and both are depicted in the figure. Extending this for cases with an unknown solution structure, the low computational cost associated with a polynomial cost function can enable the identification of the global minimum by conducting a survey of many initial guesses to identify all local minima. Finally, having identified the minimum total $\Delta V$ using the optimizer, the surrogate can be used to solve for the $x$-, $y$-, and $z$-components for each of the two burns, and these solutions can be used as initial guesses for high-fidelity optimization tools.

6. Conclusions

This paper applied the technique of surrogate modeling to maneuver design in the Earth-Moon three-body system. Specifically, the method of least squares regression was used to model the $\Delta V$ required for mid-field rendezvous of spacecraft in the Earth-Moon CRTBP, in both a halo orbit and a DRO. Additional savings in the number of samples required to generate the models were seen by using an asymptotic sampling measure, as opposed to the natural orthogonal sampling measures associated with the basis functions, resulting in the need for on the order of $10^2$ training samples for the
case of three input design dimensions and fewer than 100 samples for the case of 2 design dimensions. Further, it was demonstrated that these surrogate models can be built with no \textit{a priori} knowledge on either the expansion order or the sample size required for convergence. Once the surrogate was generated, the coefficients were used for an analytic global sensitivity analysis, and, finally, the model was shown to be ideally suited for use in evaluating the cost function for efficient maneuver optimization. The cases of rendezvous in a halo orbit and a DRO demonstrated the superiority of surrogate modeling over traditional Monte Carlo approaches in analysis capabilities and computational cost, making it a useful tool for maneuver design and analysis in the CRTBP.

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