

**Motion Relative to a Non-Inertial Frame:  
Coriolis and Centripetal Acceleration**

**ASEN 3200**

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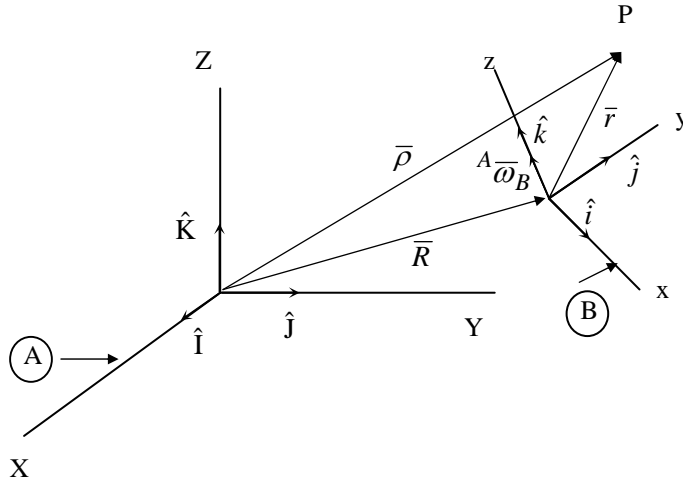


Figure 1. Definition of the (A) and (B) frame

(A)  $\equiv$  Inertial frame

(B)  $\equiv$  Non-inertial frame that is accelerating and rotating ( $\ddot{\bar{R}}, \bar{\omega}$  and  $\dot{\bar{\omega}} \neq 0$ )

Assume that we have an inertial frame, (A), in which Newton's second law applies. We now wish to describe the motion of particle P in the non-inertial frame, (B).

We wish to determine the equations of motion relative to the (B) frame using the fact that we can apply Newton's 2<sup>nd</sup> law in the (A) frame. From Fig. 1,

$$\bar{\rho} = \bar{R} + \bar{r} \quad (1)$$

$${}^A \dot{\bar{\rho}} = {}^A \dot{\bar{R}} + {}^A \dot{\bar{r}}. \quad (2)$$

Where the notation  ${}^A ( )$  means the vector as seen by an observer in the (A) frame e.g.,  ${}^A \dot{\bar{\rho}}$  is the velocity of P relative to the (A) frame. We can relate the velocity in the (A) frame to that in the (B) frame by using,

$${}^A \dot{\bar{r}} = {}^B \dot{\bar{r}} + {}^A \bar{\omega}_B \times \bar{r}. \quad (3)$$

Where  ${}^A \bar{\omega}_B$  is the angular velocity of the (B) frame as seen by an observer in the (A) frame, and  ${}^B \dot{\bar{r}}$  is the velocity of the particle P as seen by an observer fixed in the (B) frame.

Equation (3), known as the velocity rule, can be considered as the definition of the derivation of a vector relative to a rotating frame. Hence, Eq. (2) can be written

$${}^A \dot{\bar{\rho}} = {}^A \dot{\bar{R}} + {}^B \dot{\bar{r}} + {}^A \bar{\omega}_B \times \bar{r}. \quad (4)$$

Differentiating Eq. (4) yields

$${}^A \ddot{\bar{\rho}} = {}^A \ddot{\bar{R}} + \frac{{}^A d {}^B \dot{\bar{r}}}{dt} + {}^A \dot{\bar{\omega}}_B \times \bar{r} + {}^A \bar{\omega}_B \times {}^A \dot{\bar{r}}. \quad (5)$$

From the velocity rule we have,

$$\frac{{}^A d {}^B \dot{\bar{r}}}{dt} = {}^B \ddot{\bar{r}} + {}^A \bar{\omega}_B \times {}^B \dot{\bar{r}} \quad (6)$$

where  ${}^B \ddot{\bar{r}} \equiv$  acceleration of P relative to the (B) frame.

Substituting Eqs. (3) and (6) into Eq. (5),

$${}^A \ddot{\bar{\rho}} = {}^A \ddot{\bar{R}} + {}^B \ddot{\bar{r}} + {}^A \bar{\omega}_B \times {}^B \dot{\bar{r}} + {}^A \dot{\bar{\omega}}_B \times \bar{r} + {}^A \bar{\omega}_B \times ({}^B \dot{\bar{r}} + {}^A \bar{\omega}_B \times \bar{r}). \quad (7)$$

Rearranging terms in Eq. (7) yields the expression relating accelerations in the two frames,

$${}^A \ddot{\bar{\rho}} = {}^A \ddot{\bar{R}} + {}^A \bar{\omega}_B \times ({}^A \bar{\omega}_B \times \bar{r}) + {}^A \dot{\bar{\omega}}_B \times \bar{r} + {}^B \ddot{\bar{r}} + 2 {}^A \bar{\omega}_B \times {}^B \dot{\bar{r}}. \quad (8)$$

Equation (8) is a vector equation and as such is not tied to a coordinate frame. In fact the vector can be expressed in any reference frame; however, all vectors in the equation must be expressed in a common coordinate frame. For example if we wish to express it in the (A) frame we must write all vectors in terms of the unit vectors of that frame. The transformation between the (A) frame and the (B) frame is given by

$$\begin{bmatrix} \hat{\mathbf{I}} \\ \hat{\mathbf{J}} \\ \hat{\mathbf{K}} \end{bmatrix}_A = T_A^B \begin{bmatrix} \hat{\mathbf{i}} \\ \hat{\mathbf{j}} \\ \hat{\mathbf{k}} \end{bmatrix}_B$$

where  $T_A^B$  is the 3 x 3 matrix of direction cosines between the unit vectors in the two frames. This matrix rotates a vector from the (B) frame into the (A) frame. Because it is an orthogonal matrix,  $[T_A^B]^{-1} = [T_A^B]^T = T_B^A$ , which rotates a vector in the (A) frame into the (B) frame.

The first three terms on the right side of Eq. (8) represent the absolute acceleration of a point fixed in the (B) frame, i.e.,  ${}^B \dot{\vec{r}} = {}^B \ddot{\vec{r}} = \mathbf{0}$ .

Also,  $2 {}^A \overline{\omega}_B \times {}^B \dot{\vec{r}} \equiv$  Coriolis acceleration

$${}^A \overline{\omega}_B \times ({}^A \overline{\omega}_B \times \vec{r}) \equiv \text{Centripetal acceleration.}$$

The term  ${}^A \dot{\overline{\omega}}_B \times \vec{r}$  does not have a specific name.

In order for (B) to be an inertial frame it is required that

$${}^A \ddot{\vec{\rho}} = {}^B \ddot{\vec{r}}. \quad (9)$$

For this to be true, we must have

$${}^A \ddot{\vec{R}} = {}^A \overline{\omega}_B = {}^A \dot{\overline{\omega}}_B = \mathbf{0}. \quad (10)$$

Equation (10) means that an inertial frame may not be accelerating or rotating relative to any other inertial frame. However, it may move with constant velocity, i.e.  ${}^A \dot{\vec{R}} = \text{constant}$ .

We now wish to apply Eq. (8) to the Earth. Assume that the (B) frame is fixed in the Earth with its origin at the center of mass (cm) of the Earth and the z-axis along the Earth's spin axis, the x-

axis through the Greenwich meridian ( $0^\circ$  longitude) and the y-axis completing a right hand triad. Hence, x and y lie in the equatorial plane.

Now introduce a frame,  $\textcircled{C}$ , with its origin coincident with that of the  $\textcircled{B}$  frame but which does not rotate. Hence, its axes are fixed in direction in inertial space with X directed toward the vernal equinox (the intersection of the equatorial and ecliptic planes), Z along the Earth's spin axis and Y completing a right-hand triad. Note that the  $\textcircled{C}$  frame is not inertial because lunar-solar perturbations on the Earth result in accelerations of its cm, i.e.  ${}^A\ddot{\bar{R}} \neq 0$ . Even though it is not truly an inertial frame,  $\textcircled{C}$ , is referred to as the Earth-Centered Inertial (ECI) frame, while frame  $\textcircled{B}$  is referred to as the Earth-Centered Fixed (ECF) frame.

We now wish to write the equations of motion of the particle P with respect to the ECI and ECF frames. Using Eq. (8), we may write the equations of motion relative to the  $\textcircled{C}$  (ECI) frame ( ${}^A\bar{\omega}_C = {}^A\dot{\bar{\omega}}_C = 0$ ) as follows

$${}^A\ddot{\bar{\rho}} = {}^A\ddot{\bar{R}} + {}^C\ddot{\bar{r}}. \quad (11)$$

By differentiating Eq. (2), we have (or use  ${}^A\ddot{\bar{r}} = {}^C\ddot{\bar{r}} + {}^A\bar{\omega}_C \times \bar{r}$ )

$${}^A\ddot{\bar{\rho}} = {}^A\ddot{\bar{R}} + {}^A\ddot{\bar{r}}.$$

Hence,

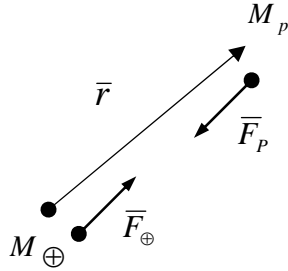
$${}^C\ddot{\bar{r}} = {}^A\ddot{\bar{r}} \quad (12)$$

and the motion of the P relative to the cm of the Earth is identical as seen in the  $\textcircled{A}$  or  $\textcircled{C}$  frame. From Newton's Second law we have (see Fig. 1)

$${}^A\ddot{\bar{\rho}} = \frac{\sum \bar{F}_p}{M_p}, \quad {}^A\ddot{\bar{R}} = \frac{\sum \bar{F}_\oplus}{M_\oplus} \quad (13)$$

where  $\oplus$  is the symbol for the Earth.

Equation (13) states that the acceleration of P as seen by an observer in a true inertial frame,  $\textcircled{A}$ , is equal to the sum of the applied forces acting on P divided by its mass, i.e. the force per unit mass. The free body diagram for the Earth and particle P can be drawn by using Newton's law of gravitation to define the gravitational forces  $\bar{F}_p$  and  $\bar{F}_\oplus$ \*



$$\begin{aligned} \sum \bar{F}_p &= -GM_p M_\oplus \frac{\bar{r}}{r^3} & \sum \bar{F}_\oplus &= GM_p M_\oplus \frac{\bar{r}}{r^3} \\ &= M_p {}^A \ddot{\bar{\rho}} & &= M_\oplus {}^A \ddot{\bar{R}} \end{aligned}$$

From Eq. (11),

$${}^C \ddot{\bar{r}} = {}^A \ddot{\bar{\rho}} - {}^A \ddot{\bar{R}} \quad (14)$$

and substituting Eqs. (13) into (14)

$${}^C \ddot{\bar{r}} = \frac{\sum \bar{F}_p}{M_p} - \frac{\sum \bar{F}_\oplus}{M_\oplus} \quad (14a)$$

$$= -G(M_\oplus + M_p) \frac{\bar{r}}{r^3}. \quad (15)$$

Because  $M_p \ll M_\oplus$  we may ignore  $M_p$  in Eq. (15) and write it as

$${}^C \ddot{\bar{r}} = -GM_\oplus \frac{\bar{r}}{r^3} = -\mu_\oplus \frac{\bar{r}}{r^3} \quad (16)$$

where,

$$\mu_\oplus = GM_\oplus. \quad (17)$$

Equation (16) is the equation of motion of P relative to the cm of the Earth as seen by an observer in the  $\textcircled{C}$  frame, and is identical to its motion as seen by an observer in the  $\textcircled{A}$  frame.

Next, we wish to write the equations of motion relative to the ECF or  $\textcircled{B}$  frame.

From Eq. (8),

$${}^B \ddot{\bar{r}} = {}^A \ddot{\bar{\rho}} - {}^A \ddot{\bar{R}} - {}^A \bar{\omega}_B \times ({}^A \bar{\omega}_B \times \bar{r}) - 2 {}^A \bar{\omega}_B \times {}^B \dot{\bar{r}}. \quad (18)$$

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\* At this point we could include the gravitational force due to other planets on  $M_\oplus$  and  $M_p$  and derive Eq. (1.2-17) of Bate, Mueller, and White.

Where we have assumed that  ${}^A\dot{\bar{\omega}}_B = 0$  for the Earth. Using Eqs. (14) and (16) to obtain

${}^A\ddot{\bar{\rho}} - {}^A\ddot{\bar{R}}$ , yields the desired results

$${}^B\ddot{\bar{r}} = -\mu_{\oplus} \frac{\bar{r}}{r^3} - {}^A\bar{\omega}_B \times ({}^A\bar{\omega}_B \times \bar{r}) - 2{}^A\bar{\omega}_B \times {}^B\dot{\bar{r}}. \quad (19)$$

Note that because the  $\textcircled{C}$  frame is not rotating,

$${}^A\bar{\omega}_B = {}^C\bar{\omega}_B.$$

In Eq. (19), we have moved the centripetal and Coriolis accelerations to the force side of the equation. In this situation they are referred to as the centripetal and Coriolis apparent forces per unit mass. Hence, the signs of the centripetal and Coriolis apparent forces per unit mass are opposite to those of the accelerations. They are referred to as “apparent” forces because they ARE NOT applied forces and should not appear on a free body diagram. These apparent forces arise only because we are describing the motion relative to a rotating coordinate frame. In other words, if we choose to solve the equations of motion in the ECI frame using Eq. (16) and then rotate the ECI frame coordinates into the ECF frame, we would never directly be aware of or have to deal with the centripetal or Coriolis forces. Moreover, the results would be identical to solving Eq. (19) directly to obtain the ECF coordinates.

If we evaluate Eq. (19) in component form using

$${}^A\bar{\omega}_B = {}^C\bar{\omega}_B = \omega \hat{k}$$

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$${}^B\dot{\bar{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k},$$

where,  $\hat{i}, \hat{j}, \hat{k}$  are unit vectors in the  $\textcircled{B}$  frame coordinate directions, the result is

$$\begin{bmatrix} {}^B\ddot{\bar{r}} \end{bmatrix}_B = \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} -\frac{\mu_{\oplus}x}{r^3} + \omega^2x + 2\omega\dot{y} \\ -\frac{\mu_{\oplus}y}{r^3} + \omega^2y - 2\omega\dot{x} \\ -\frac{\mu_{\oplus}z}{r^3} \end{bmatrix}. \quad (20)$$

Where  $\begin{bmatrix} {}^B\ddot{\bar{r}} \end{bmatrix}_B$  donates the acceleration of P relative to the  $\textcircled{B}$  (ECF) frame expressed in the  $\textcircled{B}$  frame coordinates.

Likewise in the  $\textcircled{\text{C}}$  (ECI) frame coordinates

$$\left[ {}^c \ddot{\bar{r}} \right]_C = \begin{bmatrix} \ddot{X} \\ \ddot{Y} \\ \ddot{Z} \end{bmatrix} = \begin{bmatrix} -\frac{\mu_{\oplus} X}{r^3} \\ -\frac{\mu_{\oplus} Y}{r^3} \\ -\frac{\mu_{\oplus} Z}{r^3} \end{bmatrix}. \quad (21)$$

As just stated, the solutions to Eqs. (20) and (21) are identical if the proper coordinate transformation is made between the two frames. Assume that we have solved Eq. (21) and we wish to transform the resulting position and velocity vectors from the ECI to the ECF frame as shown in Fig. 2.

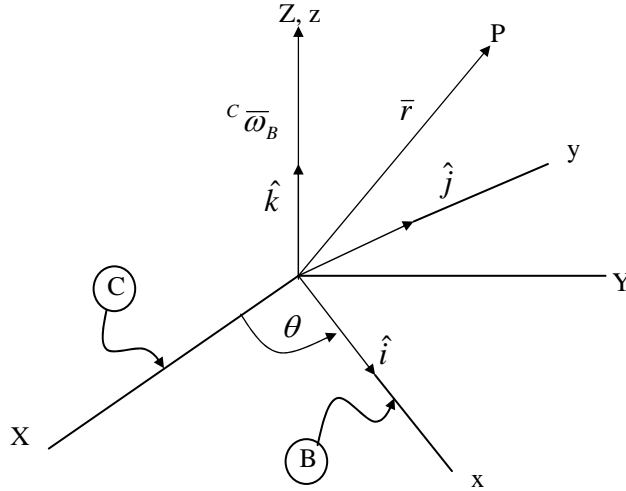


Figure 2. Orientation of the ECI,  $\textcircled{\text{C}}$ , and ECF,  $\textcircled{\text{B}}$  frames.

Note in Fig. 2 that

$$\begin{aligned} \theta &\equiv \text{Greenwich hour angle} \\ \dot{\theta} &\equiv \omega = \frac{2\pi}{23^h 56^m 04.09^s} \\ &= 7.2921158553 \times 10^{-5} \text{ rad/s.} \end{aligned}$$

The transformation of the position vector from the  $\textcircled{\text{C}}$  (ECI) frame to the  $\textcircled{\text{B}}$  (ECF) frame is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}. \quad (22)$$

This may be written as

$$[\bar{r}]_B = T_B^C [\bar{r}]_C. \quad (23)$$

Where

$$T_B^C = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (24)$$

We may transform the velocity vector by using the velocity rule

$$[{}^B \dot{\bar{r}}]_B = T_B^C [{}^C \dot{\bar{r}} - {}^C \bar{\omega}_B \times \bar{r}]_C. \quad (25)$$

In component form

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = T_B^C \begin{bmatrix} \dot{X} + \omega Y \\ \dot{Y} - \omega X \\ \dot{Z} \end{bmatrix}. \quad (26)$$

Equation (25) could also be obtained by differentiating Eq. (23), i.e.

$$[{}^B \dot{\bar{r}}]_B = T_B^C [{}^C \dot{\bar{r}}]_C + \dot{T}_B^C [\bar{r}]_C. \quad (27)$$

It can be shown that

$$\dot{T}_B^C [\bar{r}]_C = -T_B^C [{}^C \bar{\omega}_B \times \bar{r}]_C \quad (28)$$

where

$$\dot{T}_B^C = \begin{bmatrix} -\dot{\theta} \sin \theta & \dot{\theta} \cos \theta & 0 \\ -\dot{\theta} \cos \theta & -\dot{\theta} \sin \theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (29)$$

and  $\dot{\theta} = \omega$ , the rotation rate of the Earth.

The transformation from the (B) frame to the (C) frame is given by

$$\left[ \bar{r} \right]_C = T_C^B \left[ \bar{r} \right]_B \quad (30)$$

and

$$\left[ {}^C \dot{\bar{r}} \right]_C = T_C^B \left[ {}^B \dot{\bar{r}} + {}^C \bar{\omega}_B \times \bar{r} \right]_B \quad (31)$$

where

$$T_C^B = \left[ T_B^C \right]^{-1} = \left[ T_B^C \right]^T. \quad (32)$$

### Coriolis and Centripetal Force/Acceleration

We now wish to look at the Coriolis and centripetal term in detail. Equation (18), which is repeated here, is the fundamental equation describing the motion of a particle, P, in the ECF frame.

$${}^B \ddot{\bar{r}} = \frac{\sum \bar{F}_P}{M_P} - \frac{\sum \bar{F}_\oplus}{M_\oplus} - {}^C \bar{\omega}_B \times ({}^C \bar{\omega}_B \times \bar{r}) - 2 {}^C \bar{\omega}_B \times {}^B \dot{\bar{r}}. \quad (33)$$

Where we have used Eq. (14a) to replace  ${}^A \ddot{\rho} - {}^A \ddot{R}$  with  ${}^C \ddot{\bar{r}}$  and used the fact that  ${}^A \bar{\omega}_B = {}^C \bar{\omega}_B$  since frame  $\textcircled{C}$  is not rotating.

If we only have the particle, P, and the Earth to deal with, we may ignore  $\sum \bar{F}_\oplus$  as we did in Eq. (16). However, when we include the sun, moon and other planets in the system, their force per unit mass on both P and the Earth must be included. Also, if we include the effects of drag, gravity anomalies, solar pressure, etc. on P, their forces per unit mass must be included in  $\frac{\sum F_P}{M_P}$ .

In Eq. (33) the Coriolis and centripetal acceleration terms are on the force side of the equation. Hence, they represent “apparent” forces per unit mass. We will now refer to them as the Coriolis force and the centripetal force and it is understood that these are forces per unit mass and consequently, have units of acceleration. Let’s examine these forces in detail.

### Coriolis Force

The Coriolis force is given by  $-2 {}^C \bar{\omega}_B \times {}^B \dot{\bar{r}}$  where  $\textcircled{B}$  is the ECF and  $\textcircled{C}$  is the ECI frame. If P has a velocity vector relative to the ECF frame, which is not parallel to the angular velocity vector of the ECF frame, P will experience a Coriolis force.

Using the right hand rule, we see that if an observer imagines him/herself facing in the direction of  ${}^B \dot{\vec{r}}$ , the Coriolis force will always be to the right of the observer in the northern hemisphere and to the left of the observer in the southern hemisphere.

### Centripetal Force

The centripetal force is given by  $-{}^C \bar{\omega}_B \times ({}^C \bar{\omega}_B \times \bar{r})$ <sup>1</sup>. Assume that the particle P is in the x-z plane of the ECF frame as shown in Fig. 3. The centripetal force vector, whose magnitude is  $\omega^2 r \cos \phi$ , will always be perpendicular to and directed outward from the spin axis. The only time the centripetal force will vanish is when P is on the spin axis.

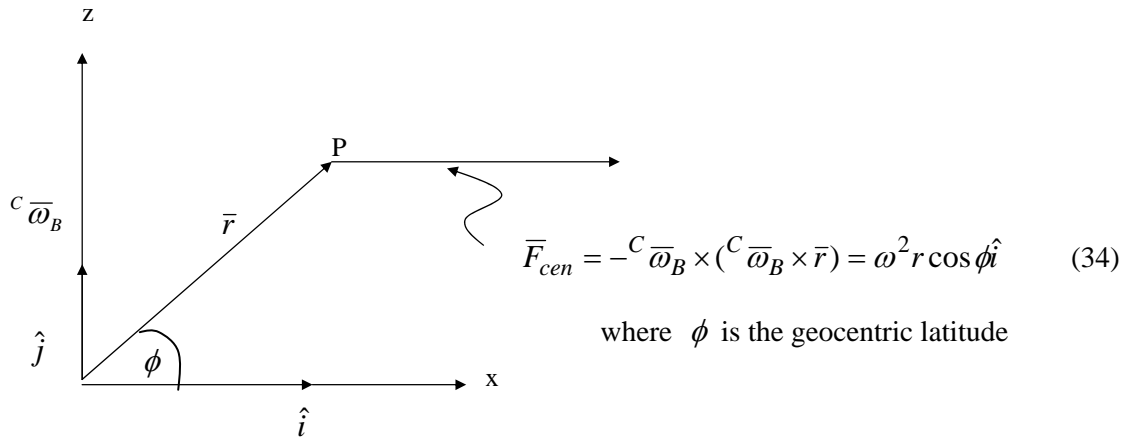


Figure 3. Centripetal Force

### Example of Coriolis and centripetal force

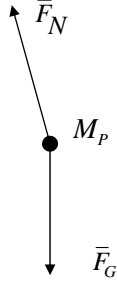
**Case 1** : P is lying at rest on the surface of a spherical Earth rotating with constant angular velocity. In this case  ${}^B \dot{\vec{r}} = {}^B \ddot{\vec{r}} = 0$  and Eq. (33) becomes

$${}^B \ddot{\vec{r}} = 0 = \frac{\sum \bar{F}_p}{M_p} - {}^C \bar{\omega}_B \times ({}^C \bar{\omega}_B \times \bar{r}) \quad (35)$$

i.e. the Coriolis force vanishes and there is a balance between the applied forces on P and the centripetal force.

<sup>1</sup> In the case of a particle traveling on a constrained circular path with constant speed as in the next example (case 1), there is a normal force exerted on the particle by the constraint. Some dynamics texts refer to this force as the centripetal force and the equal but opposite force exerted on the constraint by the particle as the centrifugal force. Using this convention the force we are calling centripetal would actually be the centrifugal force.

The free body diagram of P shows the applied forces includes the gravitational attraction of the Earth,  $\bar{F}_g$ , and the reaction force of the Earth,  $\bar{F}_N$ .



$\bar{F}_G$  and  $\bar{F}_N$  are given by

$$\bar{F}_G = -GM_{\oplus}M_p \frac{\bar{r}}{r^3} = -\mu_{\oplus}M_p \frac{\bar{r}}{r^3} \quad (36)$$

$\bar{F}_N$  = Reaction force exerted on  $M_p$  by the Earth

Substituting into Eq. (35)

$$-\mu_{\oplus} \frac{\bar{r}}{r^3} + \frac{\bar{F}_N}{M_p} - {}^C \bar{\omega}_B \times ({}^C \bar{\omega}_B \times \bar{r}) = 0. \quad (37)$$

Hence,  $\bar{F}_N$  is given by

$$\bar{F}_N = M_p \left[ \mu_{\oplus} \frac{\bar{r}}{r^3} + {}^C \bar{\omega}_B \times ({}^C \bar{\omega}_B \times \bar{r}) \right]. \quad (38)$$

Let's look at the components of the centripetal force in Eq. (37).

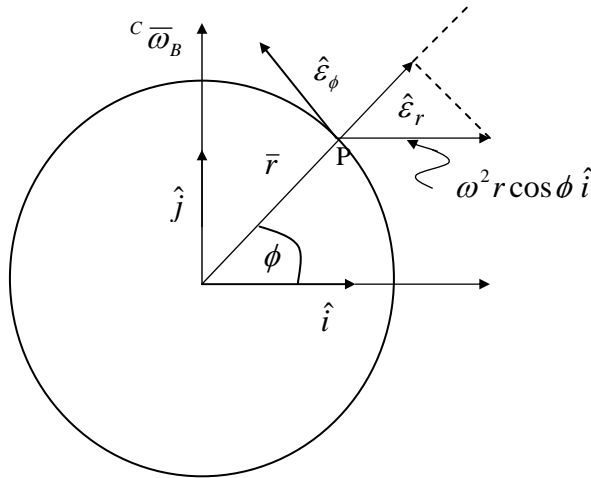


Figure 4. Components of centripetal force

As illustrated in Fig. 4,

$$-{}^C\bar{\omega}_B \times ({}^C\bar{\omega}_B \times \bar{r}) = \omega^2 r \cos^2 \phi \hat{\mathbf{e}}_r - \omega^2 r \cos \phi \sin \phi \hat{\mathbf{e}}_\phi. \quad (39)$$

Hence, Eq. (38) becomes

$$\bar{F}_N = M_P \left[ \left( \frac{\mu}{r^2} - \omega^2 r \cos^2 \phi \right) \hat{\mathbf{e}}_r + \omega^2 r \cos \phi \sin \phi \hat{\mathbf{e}}_\phi \right]. \quad (40)$$

The weight of the mass  $M_P$  is the radial component of Eq. (40).

For example, at the poles the centripetal force vanishes and

$$\bar{F}_N = M_P \left( \frac{\mu_\oplus}{r^2} \right) \hat{\mathbf{e}}_r. \quad (41)$$

While at the equator

$$\bar{F}_N = M_P \left[ \frac{\mu_\oplus}{r^2} - \omega^2 r \right] \hat{\mathbf{e}}_r. \quad (42)$$

Evaluating Eqs. (41) and (42) for  $r=6378$  km and  $\mu_\oplus = 398600 \frac{km^3}{s^2}$ ,

at the poles yields,

$$\bar{F}_N = 9.8 M_P \hat{\mathbf{e}}_r \quad \text{Newtons} = kg \frac{m}{s^2}.$$

At the equator,

$$\bar{F}_N = 9.76 M_P \hat{\mathbf{e}}_r \quad \text{Newtons.}$$

Hence, a person who weighted 200 lbs at the North Pole would weigh

$$\frac{9.76}{9.8} (200) = 199.3 \quad \text{lbs}$$

on the equator. We have ignored the oblateness of the Earth, which would change these numbers slightly because of variations in  $\mu_\oplus$ .

Note that in order to maintain the particle P in place, the Earth must exert a force in the  $\hat{\mathbf{e}}_\phi$  direction equal to  $\omega^2 r \cos \phi \sin \phi$  (see Eq. 40). Neglecting friction, the Earth cannot exert a force on P in the  $\hat{\mathbf{e}}_\phi$  direction; hence, centripetal force will cause P to move toward the equator.

Consequently, particles of mass constituting the Earth will move toward the equator in both hemispheres until the centripetal force is balanced by the gravitational force.

This can be illustrated as follows. Define  $\bar{F}_g$  as the sum of the Newtonian gravitational force,  $\bar{F}_G$ , and the centripetal force,  $\bar{F}_c$ , and call it the local gravitational force. Because of centripetal force the mass of the Earth will continue to move equatorward until the local gravitational force is normal to the local tangent plane to the Earth's surface. This is illustrated in Figs. 5 and 6. In Fig. 5 it is seen that for a spherical Earth, mass is shifted toward the equator by the local gravitational force. In Fig. 6 the equilibrium position is illustrated. Here the Newtonian gravitational force balances the centripetal force i. e., the local gravitational force  $\bar{F}_g$  is normal to the local tangent plane for any point on the Earth's surface.

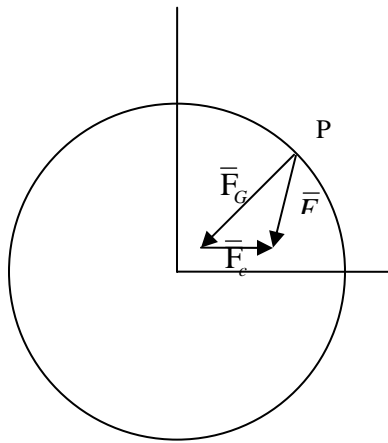


Figure 5. Forces on P for a spherical Earth

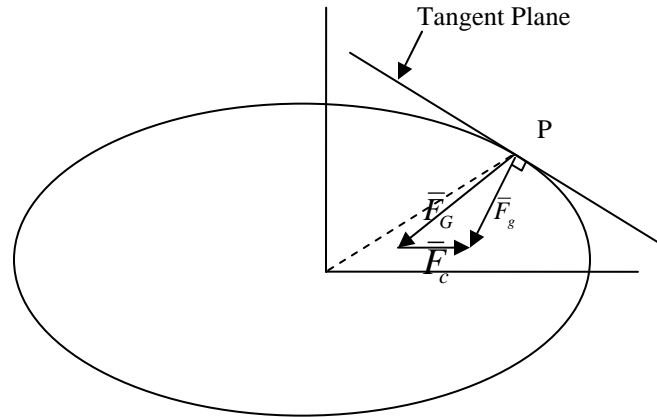


Figure 6. Forces on P for an oblate Earth

For the Earth, with a sidereal rotation period of 23h 56m 04.09s, the equilibrium oblate spheroid of revolution has an equatorial radius which is 20km greater than the polar radius. Note that if the Earth stopped rotating, the centripetal force would vanish and the Newtonian gravitational force would tend to force the Earth back to a spherical shape.

**Case 2:** Assume that P is in a geostationary orbit. Use Eq. (33) to determine its altitude. Because the orbit is geostationary it lies in the equator and is fixed in the  $\textcircled{B}$  frame. Therefore,

${}^B \dot{\bar{r}} = {}^B \ddot{\bar{r}} = 0$  and Eq. (33) becomes

$${}^B \ddot{\bar{r}} = 0 = \frac{\sum \bar{F}_p}{M_p} - {}^C \bar{\omega}_B \times ({}^C \bar{\omega}_B \times \bar{r}). \quad (43)$$

The only force acting on P is the gravitational attraction of the Earth. Hence, Eq. (43) becomes

$$-\frac{\mu_{\oplus}}{r^2} \hat{e}_r + \omega^2 r \hat{e}_r = 0, \quad (44)$$

where  $\omega$  is the angular velocity of the satellite which, for a geostationary satellite, must equal the rotation rate for the Earth. Solving Eq. (44) for  $r$  yields

$$r^3 = \frac{\mu_{\oplus}}{\omega^2} = \frac{398600}{(7.292 \times 10^{-5})^2} \frac{km^3 / s^2}{rad^2 / s^2}$$

$$r = 42,164 \text{ km.}$$

Geosynchronous altitude is

$$h = r - R = 42,164 - 6,378 = 35,786 \text{ km.}$$

As a check we can compute the period of the orbit,

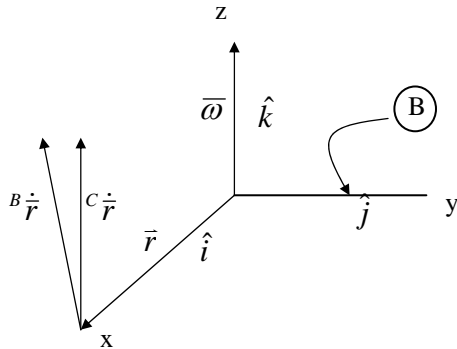
$$P = \frac{2\pi}{\mu_{\oplus}^{1/2}} r^{3/2} = 86,164 \text{ sec} = 23^h 56^m 4^s,$$

which is the sidereal rotation period of the earth.

**Case 3:** Compute the Coriolis and centripetal force per unit mass on a spacecraft in an 822 km. circular polar orbit as it flies over the equator and the South Pole.

Assume the spacecraft position vector lies along the x-axis of the  $\textcircled{B}$  (ECF) frame.

The velocity,  ${}^C \dot{\bar{r}}$ , relative to the  $\textcircled{C}$  (ECI) frame will be normal to the equator with magnitude



$$|{}^C \dot{\bar{r}}| = \sqrt{\frac{\mu_{\oplus}}{a}} = 7.44 \text{ km/s.}$$

Hence,

$${}^C \dot{\bar{r}} = 7.44 \hat{k} \text{ km/s.}$$

However, for the Coriolis force per unit mass we need the velocity relative to the ECF frame,

${}^B \dot{\bar{r}}$ , i.e.

$$\bar{F}_{Cor} = -2^C \bar{\omega}_B \times {}^B \dot{\bar{r}} = -2^C \bar{\omega}_B \times ({}^C \dot{\bar{r}} - {}^C \bar{\omega}_B \times \bar{r})$$

but

$${}^C \bar{\omega}_B \times {}^C \dot{\bar{r}} = 0.$$

Hence,

$$\begin{aligned} \bar{F}_{Cor} &= 2^C \bar{\omega}_B \times ({}^C \bar{\omega}_B \times \bar{r}) \\ &= -2\omega^2 r \hat{i} \\ &= -7.66 \times 10^{-5} \hat{i} \quad \frac{km}{s^2} \quad (\text{Inward along } \bar{r}). \end{aligned}$$

The centripetal force per unit mass is

$$\begin{aligned} \bar{F}_{cent} &= -{}^C \bar{\omega}_B \times ({}^C \bar{\omega}_B \times \bar{r}) \\ &= \omega^2 r \hat{i} \\ &= 3.83 \times 10^{-5} \hat{i} \quad \frac{km}{s^2} \quad (\text{Outward along } \bar{r}). \end{aligned}$$

As the spacecraft flies over the South Pole the centripetal force will vanish ( ${}^C \bar{\omega}_B \times \bar{r} = 0$ ).

In this case  ${}^C \dot{\bar{r}} = {}^B \dot{\bar{r}}$  and if we assume that  ${}^B \dot{\bar{r}}$  is parallel to the x axis, the Coriolis force per unit mass is

$$\begin{aligned} \bar{F}_{Cor} &= -2^C \bar{\omega}_B \times \left| {}^B \dot{\bar{r}} \right| \hat{i} \\ &= -2\omega \left| {}^B \dot{\bar{r}} \right| \hat{j} \\ &= -1.09 \times 10^{-3} \quad \frac{km}{s^2} \hat{j} \end{aligned}$$

(To the left of an observer standing on the South Pole and facing in the velocity direction).

## Summary

1. The centripetal force per unit mass will always be normal to the spin axis and directed outward. The centripetal acceleration will be equal and opposite of the centripetal force per unit mass.
2. The Coriolis force per unit mass will always be to the right of an observer in the northern hemisphere and to the left of an observer in the southern hemisphere facing in the velocity direction.