Given:

\[
X(t_{i+1}) = \begin{bmatrix} x_1(t_{i+1}) \\ x_2(t_{i+1}) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \end{bmatrix}
\]

\[
Y(t_i) = \begin{bmatrix} y_1(t_i) \\ y_2(t_i) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(t_i) \\ x_2(t_i) \end{bmatrix} + \begin{bmatrix} \varepsilon_1(t_i) \\ \varepsilon_2(t_i) \end{bmatrix}.
\]

Hence,

\[
\Phi(t_{i+1}, t_i) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}
\]

and

\[
\tilde{H}(t_i) = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{1}{2} \end{bmatrix}.
\]

Note that both the state propagation and observation-state equations are linear.

Assume the following a priori information is given:

\[
\bar{X}(t_0) = \begin{bmatrix} \bar{x}_1(t_0) \\ \bar{x}_2(t_0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad \bar{P}(t_0) = I
\]

\[
R(t_i) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ for all values of } i
\]

\[
Y(t_i) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\]
Find: $\hat{X}(t_i)$, $\hat{X}(t_0)$, $P(t_i)$, and $P(t_0)$ using the sequential (Kalman) filter. Verify the value of $\hat{X}(t_0)$ and $P(t_0)$ by using the batch processor algorithm.
Step #1 – Do a time update at \( t_1 \). Note that there is no observation at \( t_0 \). If this were the case we would first do a measurement update.

\[
\bar{X}(t_1) = \Phi(t_1,t_0) \bar{X}(t_0)
\]
\[
= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}
\]

\[
P(t_1) = \Phi(t_1,t_0) \Phi^T(t_1,t_0)
\]
\[
= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}
\]

Step #2 – Measurement update at \( t_1 \)

\[
K(t_1) = P(t_1) \tilde{H}^T(t_1) (\tilde{H}(t_1) P(t_1) \tilde{H}^T(t_1) + R)^{-1}
\]
\[
= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]
\[
= \begin{bmatrix} 8/13 \\ 1 \end{bmatrix}
\]

\[
\dot{X}(t_1) = \bar{X}(t_1) + K(t_1) (Y(t_1) - \tilde{H}(t_1) \bar{X}(t_1))
\]
\[
= \begin{bmatrix} 5 \\ 3 \end{bmatrix} + \begin{bmatrix} 8/13 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 25/8 \\ 9/4 \end{bmatrix}
\]

\[
P(t_1) = [I - K(t_1) \tilde{H}(t_1)] P(t_1)
\]
\[
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 8/13 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 19/24 \\ 1/4 \end{bmatrix}
\]

We may now map $\dot{\mathbf{x}}(t_1)$ to $t_0$ to obtain $\dot{\mathbf{x}}(t_0)$, i.e.,

$$
\dot{\mathbf{x}}(t_0) = \Phi(t_0, t_1) \dot{\mathbf{x}}(t_1)
$$

$$
= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 25/8 \\ 9/4 \end{bmatrix}
$$

$$
= \begin{bmatrix} 7/8 \\ 9/4 \end{bmatrix}.
$$

Note that $\Phi(t_0, t_1) = \Phi^{-1}(t_1, t_0)$.

$P(t_0)$ is given by

$$
P(t_0) = \Phi(t_0, t_1) P \Phi^T(t_0, t_1)
$$

$$
= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 19/24 & 1/4 \\ 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} 19/24 & -1/4 \\ -1/4 & 1/2 \end{bmatrix}.
$$
The batch processor may be used to compute $\hat{X}(t_0)$,

$$\hat{X}(t_0) = \left( H^T(t_0)R^{-1}H(t_0) + \bar{P}_0^{-1} \right)^{-1} \left( H^T(t_0)R^{-1}Y(t_1) + \bar{P}_0^{-1}\bar{X}(t_0) \right)$$

where

$$H(t_0) = \tilde{H}(t_1)\Phi(t_1,t_0)$$

$$= \begin{bmatrix} 0 & \frac{1}{2} & 1 & 1 \\ 1 & \frac{1}{2} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & \frac{3}{2} \end{bmatrix}$$

and

$$\hat{X}(t_0) = \left( \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} \left( \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} \frac{7}{8} \\ \frac{9}{4} \end{bmatrix}.$$ 

and

$$P(t_0) = \left( \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} \frac{19}{24} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{2} \end{bmatrix}.$$ 

Hence, the values of $\hat{X}(t_0)$ and $P(t_0)$ from the batch processor agree with the results of the Kaman or sequential filter.