ASEN 5070
EXAM No. 1
October 4, 2004
Solution to Problem # 1

I. (50%) The state vector is given by $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where $x_1$ and $x_2$ are constants. An 

*a priori* value $X^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$ is given with its relative accuracy described by the 

matrix, $W$. Three observations of the state are given by 

$y_1 = x_1^2 + x_2 + \varepsilon_1$

$y_2 = 2x_1 + 2x_2^2 + \varepsilon_2$

$y_3 = x_1^3 + x_2^3 + \varepsilon_3$

The relative accuracy of the three observations is given by the weighting matrix, $W$.

1. Set up a step by step algorithm describing how you would solve for the 
weighted least squares estimate, $\hat{X}$, including use of the *a priori* information. 
Define the $\Phi$, $H$, and $\Phi$ matrices (i.e., what is each element of the matrices).

Answer the following questions:

2. Is the observation-state relationship linear or nonlinear?

3. Was it necessary to use both a state and observation deviation vector?

4. Do the $\tilde{H}$, $H$, and $\Phi$ matrices differ for this problem? Why or why not?

5. Was it necessary to generate a computed observation to solve this problem? Why or why not?

**Solution algorithm:**

The observation-state relationship is nonlinear; hence, both a state and observation deviation vector must be used. Define

$y = Y - Y^*$ \text{ i.e. the observed minus the computed value of the observation vector,}

where

$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ and $Y^* = \begin{bmatrix} x_1^2 + x_2 \\ 2x_1 + 2x_2^2 \\ x_1^3 + x_2^3 \end{bmatrix}^*$,
i.e. \( Y \) is the value of the actual observations and \( Y^* \) is the computed observation based on the nominal or reference value, \( X^* \) (in this case the a priori value). The state deviation vector is defined by

\[
x = X_T - X^*
\]

where \( X_T \) is the true value of the state vector and \( X^* \) is the nominal (a priori) value.

The matrix \( \tilde{H} \) is computed from

\[
Y = G(X^*) + \varepsilon
\]

and

\[
\tilde{H} = \frac{\partial G(X^*)}{\partial X} = \begin{bmatrix}
\frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\
\frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \\
\frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2}
\end{bmatrix} = \begin{bmatrix}
2x_1 & 1 \\
2 & 4x_2 \\
2x_1 & 3x_2^2
\end{bmatrix}.
\]

Where \([\ ]^*\) indicates that \( \tilde{H} \) is evaluated on the reference value of \( X^* \).

Because \( X \) is a constant vector, \( \Phi(t_{t_0}) = I \) and \( \tilde{H} = H \).

The least squares equation may be used to compute \( \hat{x} \)

\[
\hat{x} = (H^TWH + W)^{-1}(H^TWy + W\bar{x})
\]

where

\[
\bar{x} = X_T - X^*, \text{ and } X_T \text{ is the true, but unknown, value of the state.}
\]

Because we are using our best information for \( X^* \), \( \bar{x} = 0 \) and our first estimate for \( \hat{x} \) is

\[
\hat{x}_1 = (H^TWH + W)^{-1}(H^TWy)
\]

After computing \( \hat{x}_1 \), a new value of \( \hat{X}^* \) is given by

\[
\hat{X}^*_1 = X^* + \hat{x}_1
\]
New values of $y$ and $H$ are now computed based on $\hat{X}_2^*$. The a priori information, $X^*$ and $\overline{W}$, should be maintained in subsequent iterations as described by Eq. (4.6.4) of the text; hence,

$$\bar{x}_2 = \bar{x}_1 - \hat{x}_1 = -\hat{x}_1,$$

since $\bar{x}_1 = 0$.

A new value of $\hat{x}$ is now computed

$$\hat{x}_2 = (H^TWH + \overline{W})^{-1}(H^Ty + \overline{W}x_2).$$

Then

$$\hat{X}_2^* = \hat{X}_2^* + \hat{x}_2$$

and

$$\bar{x}_3 = \bar{x}_2 - \hat{x}_2$$

Next new values of $y$ and $H$ are computed and $\hat{x}_3$ is computed.

This procedure is continued until the process has converged, i.e. $|\hat{x}| \Rightarrow 0$.

Convergence occurs when the observations are balanced by the a priori information. Neither $y$ nor $\bar{x}$ will be zero but they will converge to values such that $(H^TWy + \overline{W}x) = 0$; thus $\hat{x} = 0$.

The best estimate of $X_f$ is the final value of $\hat{X}_f^*$.

Answers to questions

1. The observation - state relation is nonlinear
2. Both a state and observation deviation vector is needed
3. The $H$ and $\tilde{H}$ matrices are identical since $\Phi = I$.
4. Anytime an observation deviation vector is needed (i.e., the observation – state or the equations of motion for the state are nonlinear) an observation residual must be computed.