I. (50%) The state vector is given by \( X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \), where \( x_1 \) and \( x_2 \) are constants. An a priori value \( X^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \) is given with its relative accuracy described by the matrix, \( W \). Three observations of the state are given by
\[
\begin{align*}
y_1 &= x_1^2 + x_2 + \epsilon_1 \\
y_2 &= 2x_1 + 2x_2^2 + \epsilon_2 \\
y_3 &= x_1^2 + x_2^3 + \epsilon_3
\end{align*}
\]
The relative accuracy of the three observations is given by the weighting matrix, \( W \).

1. Set up a step by step algorithm describing how you would solve for the weighted least squares estimate, \( \hat{X} \), including use of the a priori information. Define the \( \Phi \), \( H \), and \( H \) matrices (i.e., what is each element of the matrices).

Answer the following questions:
2. Is the observation – state relationship linear or nonlinear?
3. Was it necessary to use both a state and observation deviation vector?
4. Do the \( \Phi \), \( H \), and \( H \) matrices differ for this problem? Why or why not?
5. Was it necessary to generate a computed observation to solve this problem? Why or why not?

II. (20%) Given the system
\[
\dot{X} - 3X^2 - 4X = 0
\]
With the state vector defined by \( X = \begin{bmatrix} X \\ \dot{X} \end{bmatrix} \), and the state deviation vector defined by \( \delta X = \begin{bmatrix} \delta X \\ \delta \dot{X} \end{bmatrix} \). Where \( \delta \) indicates a small deviation from a reference value.

a. Write the linearized equations in state space form, \( \delta \dot{X} = A \delta X \).

b. How would you determine the state transition matrix for this system? What additional information is needed to generate the state transition matrix?
III. (30%) Circle the correct answer or answers.

a. Given the observation-state equation

\[
y(t_i) = t_i^2ax_0 + t_icx_i \quad i = 1, 2, \ldots, 10.
\]

Where \(a, x_0, x_i, c\) are constants and \(t_i\) is given. Which of the following state vectors are observable:

\[
\begin{bmatrix}
    a \\
    x_0 \\
    c
\end{bmatrix}
\quad \text{1.}
\quad \begin{bmatrix}
    a \\
    c
\end{bmatrix}
\quad \text{2.}
\quad \begin{bmatrix}
    a \\
    x_0
\end{bmatrix}
\quad \text{3.}
\quad \begin{bmatrix}
    x_0 \\
    c
\end{bmatrix}
\quad \text{4.}
\quad \begin{bmatrix}
    x_1 \\
    x_0
\end{bmatrix}
\quad \text{5.}
\]

b. The differential equation

\[
x\ddot{x} + ax^2 + t^4 = 0
\]

is

1. 1st order and 1st degree
2. 2nd order and 1st degree
3. linear
4. nonlinear

c. Given two observations and the second is twice as accurate as the first, we would use the following weighting matrix

\[
\begin{bmatrix}
    1 & 0 \\
    0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
    2 & 0 \\
    0 & 1
\end{bmatrix}, \quad \begin{bmatrix}
    1 & 0 \\
    0 & 2
\end{bmatrix}, \quad \begin{bmatrix}
    2 & 1 \\
    0 & 1
\end{bmatrix},
\]

d. If the differential equation for the state is given by

\[
\dot{x} + bx^2 = 0
\]

It would not be necessary to use a state deviation vector. T or F

e. The state transition matrix will contain terms such as \(\frac{\partial \dot{x}(t)}{\partial J_2(t_0)}\). The units of this partial derivative are

1. L/T, 2. L²/T, 3. 1/T, 4. It is dimensionless

f. If the state transition matrix is symplectic it can be inverted by inspection. T or F